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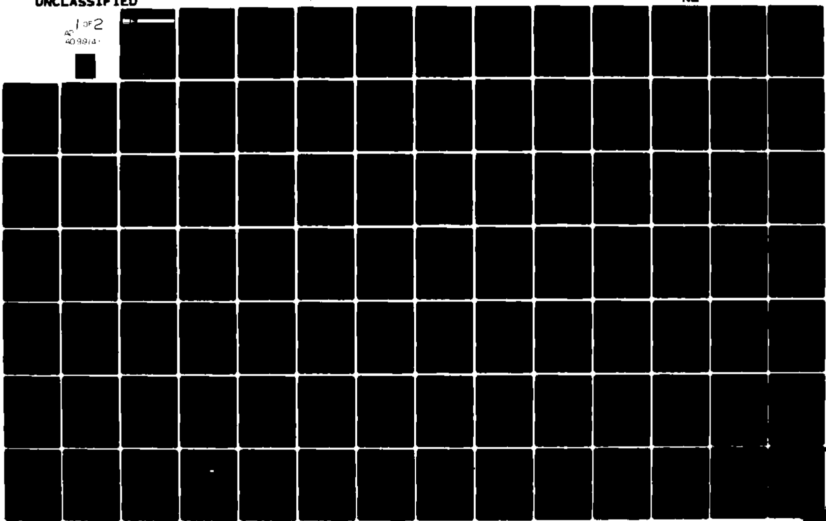

LEVEL 2

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FINAL REPORT

PARAMETER INSENSITIVE CONTROL

Project 5203

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I. INTRODUCTION AND SUMMARY

The objective of this research study is to understand more about the characteristics of control systems that are parameter insensitive. Consider the generic feedback control system as illustrated in Figure 1. The requirement on feedback control is to maintain system performance within a specified tolerable range irrespective of uncertainty in the models of the system as well as the disturbances. The reason for this requirement is that the control is designed on the basis of a nominal model of the system, whereas the control operates on the real system for which the model is only an approximation. When the true system parameters or behavior exceed those accounted for in the model the controlled system may fail to operate within the required performance envelope. Thus, the need for parameter insensitive control.

The generic problem described above gives rise to two specific problems - a sensitivity problem and a robustness problem.

The sensitivity problem is concerned with small changes in parameter values about nominal design values, and the relation of these small changes to specific system properties (e.g. stability, pole-placement, etc.) which are to be maintained within specified ranges.

The robustness problem is concerned with classes of models or sets of parameters (including the nominal class or set) for which specific system properties (like stability) are to be maintained. Clearly, robustness includes sensitivity, whereas sensitivity does not imply robustness.

The focus in this study is on robustness of linear systems particularly stability robustness and performance robustness. Stability robustness refers to the ability of the control to

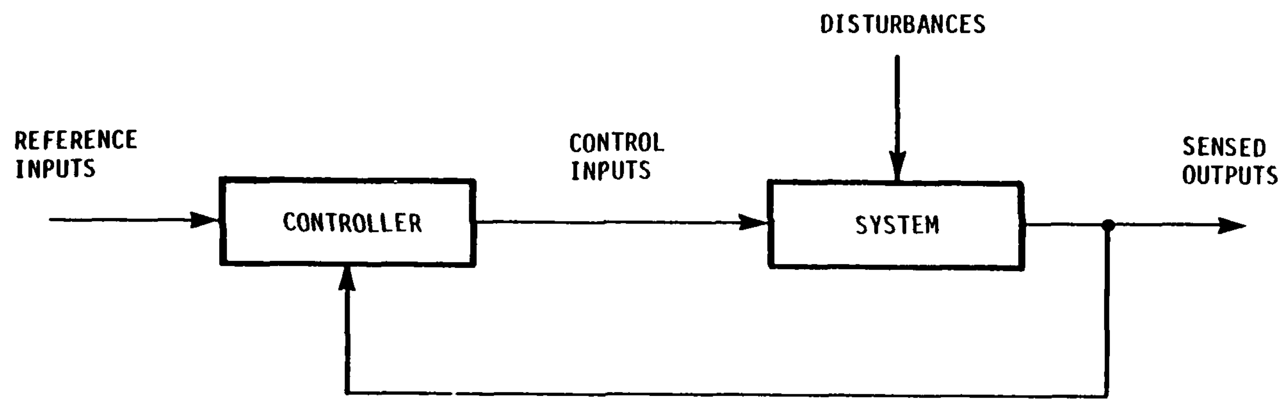

Figure 1 Generic Feedback System

maintain stability in the presence of model uncertainty. Performance robustness refers to the ability of the control to maintain a measure (or measures) of performance within a specified tolerable range. Stability robustness has received considerable attention, for example see the discussion in [1] and the detailed coverage in [2]. Performance robustness has not received the same attention, primarily due to the general lack of satisfactory results in stability robustness, and the considerably more difficult nature of the performance robustness problem. Attempts have been made in performance robustness, for example, maintaining a prescribed degree of exponential stability [3] in the presence of uncertainty. In addition, the focus has been on analysis of robustness (i.e. how robust is the control) rather than on the synthesis of robustness (i.e. the design of control with a specified measure of robustness). The ultimate aim would be a method to synthesize control laws which have specific stability robustness and performance robustness properties.

New approaches to the stability and performance robustness synthesis problems are presented in this report as well as areas for future research.

The appendices contain detailed investigations in several areas. Appendix A discusses the performance robustness issue in a very precise setting by defining a worst case design and a best average design optimization procedure. It is shown that state-feedback or observer-based feedback controllers with constant gains are not optimal when the parameters of the sytem change. An adaptive control, which is optimal, updates the gains in accordance with parameter changes by effectively using past knowledge of the states. The controllers and filters proposed here use a limited amount of past states but with constant gains. The gains are selected to either maximize worst case performance or maximize average performance.

Appendix B shows new results in the use of LQG design by extending the theory to allow for frequency-shaping in the cost functional. This extention provides a means to meet classical robustness requirements with the automated computational procedure of modern control theory. The frequency-shaping method permits the designer to synthesize an optimal compensator which takes into account a priori knowledge of disturbances and parameter uncertainty. For example, if the transfer function of the model were known to lie within a certain range of transfer functions (corresponding to a range of parameter variations) then the cost functional can be frequency shaped so that equal penalty is applied across this range. The degree of penalty must be determined iteratively by evaluating the performance. This procedure produces performance robustness to a specified range of parameter variations or disturbances. Appendix B shows four examples of applying this methodology: flexible spacecraft, helicopter vibration, industrial crane, and aircraft in lateral wind.

Appendix C approaches the robustness problem from an algebraic-geometric view. Stable subspaces are designed for a class of controls which remain stable for a specified class of parameter variations. This analysis also discriminates systems where linear state feedback control laws work adequately and systems where adaptive control laws are required.

Appendix D presents methods to design robust controllers for flexible structures. Flexible structures such as an aircraft wing, long beam, or a large space structure (LSS) are representatives from a class of systems with infinite modes. The actual dynamics are governed by partial differential equations for which the lower frequency modes can be adequately (in most cases) modeled by interconnected oscillatory systems. The difficulty arises in that the dimension of this model is extremely large (e.g., thousands of modes), and the higher modal frequencies and modes shapes are poorly known. However, the

structure of the model is definitely known--i.e., positive symmetric matrix second order equations. The problem then is to find robust controllers which are based on a lower order model of the flexible structure. Two approaches are presented. The first method (Appendix D.1) uses the solution of an inverse optimal controller problem to establish an iterative procedure to design robust co-located vibration control, for poorly damped flexible structures. The solution requires output feedback with specified constraints, leading to robustness with respect to unmodeled modes and a large class of parameter variations. The robustness properties are proved directly from known properties of control laws optimizing quadratic performance measures. The second method (Appendix D-2) uses the frequency domain robustness measures discussed in Section 3.2, to design an LQG modal control which is robust with respect to the unmodeled but bounded residual modes of the structure. The robustness measures are used to synthesize a shape/vibration LQG modal control using position measurements.

1.1 SHORT SURVEY AND FURTHER ISSUES

Section II describes some of the issues of robustness with respect to the well established LQG control. Section III surveys recently developed measures of robustness in the time-domain and in the frequency-domain. Several promising areas for future application and investigation of these robustness methods are also presented which highlight new ways to consider the issues in linearized gain scheduling control, reduced order control, the effect of unmodeled but bounded perturbations, and the effect of uncertainty and reliability of actuators and sensors.

II. ROBUSTNESS OF THE LQG CONTROL

Consider the system governed by the time-invariant linear model,

$$\dot{x} = Ax + Bu + w \ , \quad t \geq o$$
$$y = Cx + v \tag{1}$$

where x is the state, u the control, y the sensed output, and (w, v) the dynamic and measurement disturbances, respectively. The disturbances (w, v) are usually assumed to be zero-mean, white, Gaussian processes with known constant intensities. For purposes here it is sufficient to assume simply that they are bounded* and may have known spectra. Since y is the available measurement, the control is restricted to the form,

$$u = -H_c y \tag{2}$$

where H_c is a linear, time-invariant operator with rational transfer function $H_c(s)$. The feedback structure is shown in Figure 2.

It is well established [4] that the LQG control,

$$u = -F\hat{x}$$
$$\dot{\hat{x}} = A\hat{x} + Bu + K(y - C\hat{x}) \tag{3}$$

stabilizes the system provided that (v, w) are bounded and the (F, K) gain matrices are selected so that the eigenvalues of both $A-BF$ and $A-KC$ all have negative real parts. If (A, B) is controllable and (A, C) is observable then the eigenvalues of $A-BF$ and $A-KC$ can be arbitrarily assigned [4]. Stability is also guaranteed if (F, K) are computed from

$$F = B^T P \ , \ K = SC^T \tag{4}$$

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* The expression "x is bounded" as used here means, $\int_0^\infty x^T(t)\, x(t)\, dt < \infty$, i. e. x is in L_2, unless otherwise stated.

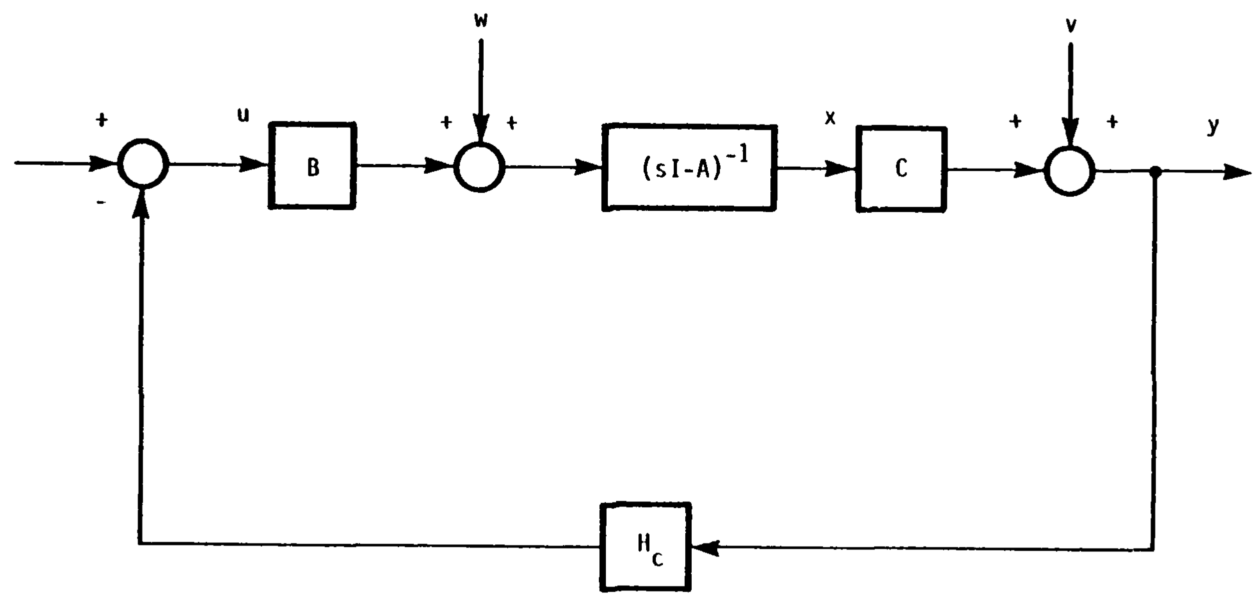

Figure 2 Linear Control

where P and S are the symetric positive definite solutions of the algebraic Ricatti equations,

$$A^TP + PA - PBB^TP + Q = 0 \tag{5}$$

$$AS + SA^T - SC^TCS + W = 0 \tag{6}$$

provided that $Q = Q^T > 0$ and $W = W^T > 0$.

For the LQG control,

$$H_c = F(sI-A_c)^{-1}K \tag{7}$$

where

$$A_c = A - BF - KC \tag{8}$$

The source of the robustness problems of LQG control can be simply stated: <u>There is no guarantee that the eigenvalues of the LQG compensator H_c are stable</u>.

Reference 5 gives an excellent description of this problem. The results of instability of the compensator H_c can be disastrous. Consider what happens if a control loop or a sensor becomes disabled! The system will most likely be unstable. Also, changes in the true system model (A, B, C) matrices may cause instability. This results because no measure of robustness has been intentionally designed into the system.

However, it has been shown [4,6,7] that the full state feedback LQ control,

$$u = -Fx \tag{9}$$

with F from (4)-(5), guarantees stability for 1/2 to infinite gain margin per channel, or $\pm$ 60^o phase margin per channel. Unfortunately, this guarantee is too restrictive in regard to robustness and other practical considerations of multi-loop control. First of all, LQ control requires full state feedback into all input channels. Generally this is impossible, since all the states are not available for measurement. Secondly, the closed-loop system with LQ control rolls off at the high

frequencies at 6 db/octave. Thus, any unmodeled high frequency phenomena in the system can have a deleterious effect on performance and stability. Thirdly, and most important to robustness, most parameter variations do not appear in the form of gain or phase perturbations. Thus, the gain and phase margin properties, attributable to the LQ control, are not valid measures of robustness.

An even more distressing situation arises when utilizing observer-based LQG control [3]. In this case, there are no guaranteed margins at all! However, the observer gain K can be adjusted [8], under certain conditions, so that the gain and phase margins of the LQG control approach asymptotically the gain and phase margin properties of the LQ control. But this does not mitigate the robustness problem since gain/phase margins, as already mentioned, are not necessarily measures of robustness.

However, the gain/phase margin properties of the LQ control result from a more general consideration discussed in [7]. Consider the perturbed LQ control shown in Figure 3. The perturbation L is assumed to be a bounded linear, time-invariant, operator with rational transfer function L(s). Under these conditions if

$$L(j\ ) + L^{T}(-j\ ) \quad I, \qquad 0 \qquad (10)$$

then the perturbed LQ control is stable. The gain and phase margin properties of LQ control can be derived from (10). However, (10) allows for a broader class of uncertainties, for example, perturbations in the B matrix of (1). For perturbations in A and C of (1), a more general measure of robustness is required.

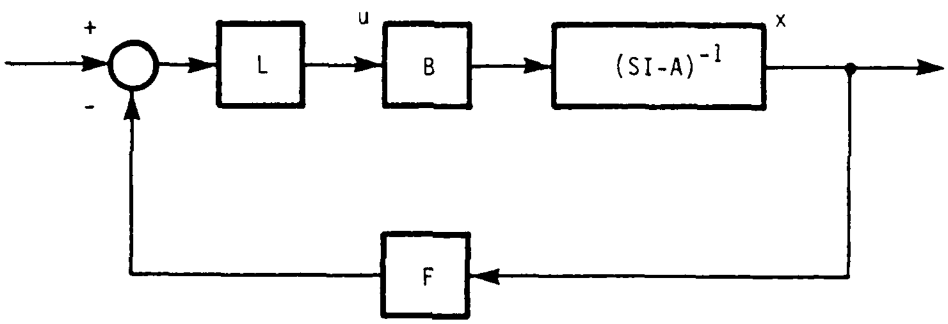

Figure 3 Perturbed LQ Control

III. MEASURES OF ROBUSTNESS

Perhaps the best and most inclusive measures of robustness are those pioneered by Nyquist [9] and Bode [10] for the single-input, single-output control system. For example, the effects of plant parameter variations or unmodeled high frequency phenomena are immediately apparent in the frequency domain using Nyquist plots. The gain and phase margins can be quantitatively established by inspection. If these margins are not adequate with a given control structure, then various combinations of lead/lag compensation networks can be used to adjust the margins. However, as soon as there is more than one input/output pair, the simple graphic understanding of the Nyquist plot vanishes in a puff of dimensionality, and the problem of robustness becomes more difficult. As was seen by the previous discussion of LQG control, the robustness problem remains, even with the move away from the frequency domain and the introduction of state-space and optimal control by Kalman [11]. Of course the linear-optimal time-domain approach does yield a stable control for most multivariable systems, provided relatively mild conditions are satisfied (like controllability and observability.)

Multivariable control can also be designed in the frequency-domain. The approaches in references [12], [13] require the concept of diagonal dominance. The control is designed in two parts. First, the system is diagnalized (abstractly) and secondly the "classical" control concepts are applied to the independent diagonal loops. This is a cumbersome approach, but even so, there is no guarantee of robustness since the actual control is not diagonal (there are transformations into and out of the "diagonal" space).

The choice of designing the control in the frequency-domain (the so called "classical" approach) or in the time-domain (the so called "modern" approach) is all wrapped up with the question of robustness. The effect of uncertainty, which the designer knows is "out there" may be more describable in either the time-domain or the frequency-domain. Attempts to measure and test robustness have been proposed for both the time-domain and the frequency-domain.

3.1 TIME-DOMAIN MEASURES OF ROBUSTNESS

Methods have been developed to measure stability robustness in the time-domain [14]. Consider the system,

$$\dot{x} = (H + \delta H)x \tag{10}$$

where H is stable and δH is a perturbation in the model. The system (10) is stability robust if and only if the eigenvalues of $H + \delta H$ all have negative real parts. This requires specific knowledge of δH which may not be available. For example, only $||\delta H||$ may be known. The approach in [14] is to form the Lyapunov function,

$$V = x^T P x \tag{11}$$

where $P = P^T > 0$ and $Q = Q^T > 0$ satisfy,

$$H^T P + PH + Q = 0 \tag{12}$$

and

$$\begin{aligned}\dot{V} &= -x^T Q\, x + 2\, x^T P\, Hx \\ &\le -||x||^2 (-\underline{\lambda}(Q) + 2\overline{\lambda}(P)\, ||\delta H||)\end{aligned}$$

is always negative if,

$$||\delta H|| < \frac{\underline{\lambda}(Q)}{2\,\overline{\lambda}(P)} \tag{13}$$

Consequently if (13) is satisfied then V is a Lyapunov function and so (10) is asymptotically stable. (The notation $\underline{\lambda}(\cdot)$ and $\overline{\lambda}(\cdot)$ represent the minimum and maximum eigenvalues, respectively, of the matrix argument.)

Consider the perturbed system of (1),

$$\dot{x} = (A + \delta A)x + (B + \delta B)u + w$$

$$y = (C + \delta C)x + v \tag{14}$$

with LQG control (3),

$$u = -F\hat{x}$$

$$\dot{\hat{x}} = A\hat{x} + Bu + K(y - C\hat{x})$$

The matrices in (10) are then,

$$H = \begin{bmatrix} A-BF & BF \\ 0 & A-KC \end{bmatrix} \tag{15}$$

$$\delta H = \begin{bmatrix} \delta A-\delta BF & \delta BF \\ \delta A-K\delta C-\delta BF & \delta BF \end{bmatrix}$$

The robustness test (13) can now be applied using the above (15) quantities. Clearly (13) can be used to test a wide variety of parameter variations, like those in (14). There are limitations. For example:

(1) The state representation restricts the form of the perturbation (e.g., unmodeled high frequency phenomena can not be accounted for in (14), without extending the state order.)

(2) If the state order is too large, the Lyapunov equation (12) may not be solvable numerically.

(3) It is not clear how to modify the control if the test fails. In fact, failure of the test does not imply instability. The test (13) is sufficient for stability.

A similar test exists for systems with non-linear perturbations,

$$\dot{x} = Hx + h(x) \tag{16}$$

where now,

$$||h(x)|| < \frac{\lambda(Q)}{2\bar{\lambda}(P)} ||x|| \tag{17}$$

insures asymptotic stability of (16).

Tests (13) or (17) can also provide a measure of performance robustness. If (P,Q) satisfy,

$$(H + \alpha I)^T P + P(H + \alpha I) + Q = 0$$

with $\alpha > o$, then (13) or (17) implies that (10) or (16) is guaranteed to be stable with degree of exponential stability equal to α. In other words, the nominal system with no perturbation exhibits performance as measured by the α-degree of exponential stability, and the system will be performance robust (in the sense of exponential stability) if the perturbations satisfy (13) or (17).

3.1.1 Linearized Control

An interesting potential application of (17) arises from the use of "linearized" control law mechanizations. A common design procedure (e.g. flight control) to control non-linear systems is to linearize about several equilibrium conditions, develop the linear control for each operating point, and then during actual operation schedule the appropriate linear gains. The real issue with this approach is in determining a minimum number of equilibrium conditions in the dynamic envelope and a gain scheduling procedure which maintains a specified degree of performance and stability throughout the dynamic operational envelope. Consider the non-linear dynamic system,

$$\dot{x} = f(x, u)$$

Define $(\bar{x}, \bar{u})$ as an equilibrium point such that

$$o = f(\bar{x}, \bar{u})$$

In general there are an infinite number of such equilibria. Suppose one candidate equilibrium has been selected. Define the errors,

$$\delta x = x - \bar{x}$$

$$\delta u = u - \bar{u}$$

consequently, by linearization about (x, u),

$$\dot{x} = Ax + Bu + e$$

where

$$A = \frac{\partial f}{\partial x}\Bigg|_{\substack{x=\bar{x}\\ u=\bar{u}}} \qquad B = \frac{\partial f}{\partial u}\Bigg|_{\substack{x=\bar{x}\\ u=\bar{u}}}$$

Select the control

$$\delta u = -F\,\delta x$$

where $A-BF$ is stable. A fundamental problem is to determine a bound on $\|\delta x\|$ to insure stability of

$$\delta\dot{x} = (A-BF)\delta x + e$$

Applying (17) gives,

$$\|e\| < \frac{\sigma(Q)}{2\bar{\sigma}(P)}\,\|\delta x\|$$

where

$$(A-BF)^T P + P(A-BF) + Q = 0$$

and

$$e = f(\bar{x} + \delta x, \bar{u} - F\delta x) - (A-BF)\,\delta x$$

The problem remains to evaluate $\|e\|$. For many non-linear systems, the non-linearities are at most 2nd degree. Thus,

$$\|e\| \le \alpha\,(F, \bar{x}, \bar{u})\|\delta x\|^2$$

where $\alpha\,(\cdot)$ is the norm of the 2nd order tensor in the Taylor series expansion of $F(\bar{x} + \delta x, \bar{u} - F\delta x)$. Consequently

$$\|\delta x\| < \frac{\sigma(Q)}{2\alpha(F, \bar{x}, \bar{u})\bar{\sigma}(P)}$$

determines the range of stability of the "linearized" control. The above bound is only illustrative of how to use the time-domain robustness measure (17). The complete problem involves many equilibria, gain scheduling, and an observer since all the states will not be available for measurement.

3.1.2 Reduced Order Control

Another potential application of (13) is to determine the stability of a reduced order control. Consider the controllable system,

$$\dot{x} = Ax + Bu$$

$$u = -Fx$$

where $A-BF$ is stable. If $\dim(x)$ is very large then a reduced order control is desired,

$$u = -K z$$

$$z = Mx$$

where $\dim(z) < \dim(x)$. Usually M is selected by insight into the system behavior. Generally x can be partitioned so that $x = (z, q)$. In this case K is just those gains in F that multiply the truncated states z. The closed loop system is,

$$\dot{x} = (A-BKM)x$$

Adding and subtracting BFx gives,

$$\dot{x} = (A-BF)x + B(F-KM)x$$

which following (13) is stable if,

$$B(F-KM) < \frac{\lambda(Q)}{2\,\sigma(P)}$$

where (P, Q) satisfy (12) with $H = A-BF$.

3.2 FREQUENCY-DOMAIN MEASURES OF ROBUSTNESS

Consider the stable linear unity feedback system of Figure 4. Solving for e gives,

$$e = (I + G_o)^{-1} u$$

If the system is perturbed as in Figure 5, then

$$e = (I + G_o + \delta G)^{-1} u$$

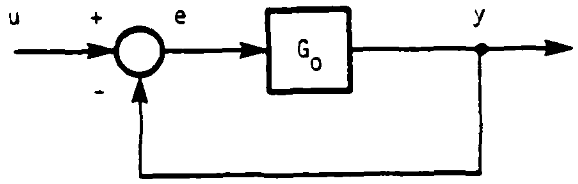

Figure 4 Unity Feedback

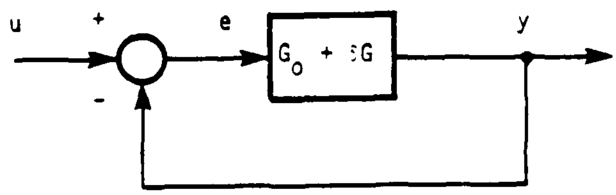

Figure 5 Perturbed Unity Feedback

From [15], [16], and [17] if u is bounded (i.e. $\|u\| < \infty$) and $(I + G_o + \delta G)^{-1}$ is bounded then the perturbed system is stable. But,

$$(I + G_o + \delta G)^{-1} = (I + (I + G_o)^{-1}\delta G)^{-1} (I + G_o)^{-1}$$

Since $(I + G_o)^{-1}$ is bounded (nominal system is stable) then stability is insured if

$$\| (I + (I + G_o)^{-1}\delta G)^{-1} \| < \infty$$

Using the contraction principal [18] the above is bounded if

$$\| (I + G_o)^{-1}\delta G \| < 1 \tag{18}$$

This is proved rigorously in [19]. Note that (18) does not require G_o or δG to be linear, time-invariant operators.

In general, the perturbations can be classified into additive and multiplicative perturbations as shown in Figure 6.

For additive perturbations $\delta G = L$ and

$$\| (I + G_o)^{-1} L \| < 1 \tag{19a}$$

insures stability. Isolating the additive perturbation gives the more conservative condition,

$$\| L \| < 1/\| (I + G_o)^{-1} \| \tag{19b}$$

For multiplicative perturbations $\delta G = G_o\, L$ and

$$\| (I + G_o^{-1})^{-1}\, L \| < 1 \tag{20a}$$

insures stability. Isolating the multiplicative perturbation gives the more conservative condition,

$$\| L \| < 1/\| (I + G_o^{-1})^{-1} \| \tag{20b}$$

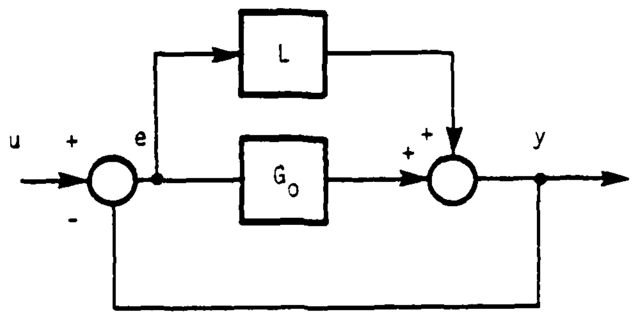

Figure 6a Additive Perturbation

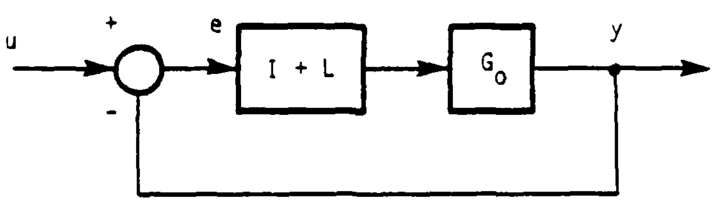

Figure 6b Multiplicative Perturbation

If G_o and L are linear, time-invariant operators with the corresponding rational transfer functions $G_o(s)$ and $L(s)$ then the perturbed system remains stable if, for additive perturbations,

$$\bar{\sigma}\Big(L(j\omega) \Big) < \underline{\sigma}\Big(I + G_o(j\omega) \Big), \quad \omega \geq 0 \tag{21}$$

or for multiplicative perturbations,

$$\bar{\sigma}\Big(L(j\omega)\Big) < \underline{\sigma}\Big(I + G_o^{-1}(j\omega)\Big), \quad \omega \geq 0 \tag{22}$$

provided that the perturbations are stable. Relations (21) and (22) are proved rigorously in [19] and also in [20] where these ideas were originally developed using Nyquist criteria arguments. Note that $\bar{\sigma}(\cdot)$ and $\underline{\sigma}(\cdot)$ represent the maximum and minimum singular values of the matrix argument.

Advantages of these frequency-domain robustness measures over their time-domain counter parts are that the matrix order of G_o is much lower then the state order, and unmodeled high frequency phenomena are easily incorporated. The disadvantage is that G_o and L are very complicated functions of a complex variable.

These tests can be applied to the LQ control. For the single-input, single-output case Kalman [6] showed that the magnitude of the return-difference for an optimal control is

$$|1 + g_o(j\omega)| \geq 1, \quad \omega \geq 0$$

Similarly for the multi-loop system Anderson and Moore [4] give the more general result,

$$\Big(I + G_o(-j\omega)\Big)^T\Big(I + G_o(j\omega)\Big) \geq I$$

Both imply that

$$\underline{\sigma}(I + G_o) \geq 1$$

which further implies that

$$\underline{\sigma}(I + G_o^{-1}) \geq 1/2$$

Consequently, LQ control can withstand additive perturbations,

$$\bar{\sigma}(L) < 1, \quad \omega \geq 0$$

or multiple perturbations,

$$\bar{\sigma}(L) < 1/2$$

and still remain stable. In general these results do not apply for the LQG control.

3.2.1 Unmodeled Perturbations

Consider the system shown in Figure 7 where

$$y = (H_p + H_R)\, u$$

$$u = H_c(v-y)$$

$$\| v \| < \infty$$

H_p is the model of the plant dynamics (sometimes referred to as the primary or nominal system model);

H_R is the residual model, where usually only $\| H_R(j\omega) \|$ is known (e.g. high frequency); and H_c is the control such that the loop is stable with $H_R=0$. Breaking the loop at u gives,

$$G = H_C\, H_P + H_C\, H_R$$

Identifying $H_C H_R$ as an additive perturbation gives

$$\bar{\sigma}(H_C H_R) < \underline{\sigma}(I + H_C H_P), \quad \omega \geq 0 \tag{23}$$

as a sufficient condition for stability. If H_c represented an LQG design then H_c would have to be stable for the test to work. If all the matrices were square then an alternate test results, since

$$G = H_C H_P(I + H_P^{-1}\, H_R)$$

In this case $H_P^{-1}\, H_R$ can be identified as a multiplicative perturbation and so,

$$\bar{\sigma}(H_p^{-1}\, H_R) < \underline{\sigma}(I + (H_c H_p)^{-1}), \quad \omega \geq 0 \tag{24}$$

insures stability.

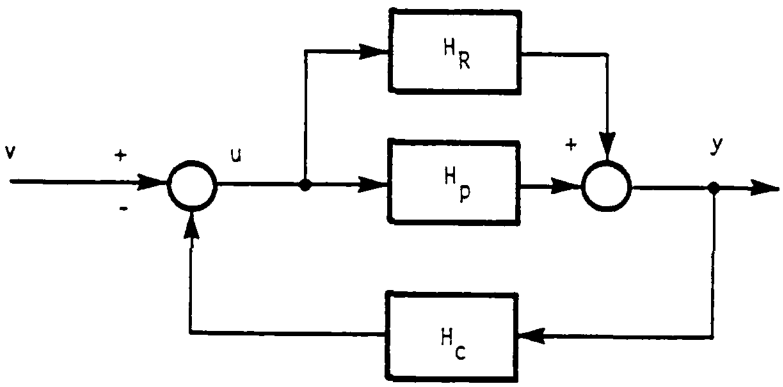

Figure 7 Effect of Residual Perturbations

In (23) if H_c is stable then the residual system H_R can be isolated to give the more conservative condition,

$$\bar{\sigma}(H_R) < \frac{\underline{\sigma}(I + H_c H_p)}{\bar{\sigma}(H_c)} \quad , \omega \geq 0 \tag{25}$$

In (24) the residual system H_R can be isolated (independent of H_c stable) to give the more conservative condition,

$$\bar{\sigma}(H_R) < \underline{\sigma}(H_p)\, \underline{\sigma}(I + (H_c H_p)^{-1}) \quad , \omega \geq 0 \tag{26}$$

Actual computations will determine which robustness measures of the above are the least conservative. Of course (26) can only be used if H_p^{-1} exists and (25) can only be used if H_c is stable.

Many systems fit the general framework of Figure 7. Flexible systems (like large space structures) have exactly this format. They are infinite dimensional systems modeled by finite dimensional systems with much lower order controllers. In this case H_R represents the effect of the unmodeled or residual modes, which are not necessarily all high frequency effects. The robustness stability of LQG modal control for systems of this type is examined in detail in Appendix D.2 using the frequency domain robustness measures discussed here.

3.2.2 Reduced Order Control

The stability of the reduced order control discussed in Section 1.3.2 can also be examined using frequency domain robustness measures. For that system,

$$G = KM\Phi B \ , \ \Phi = (sI - A)^{-1}$$

Adding and subtracting $F\Phi B$ gives,

$$G = F\Phi B - (F-KM)\Phi B$$

and so

$$\bar{\sigma}((F-KM)\Phi B) < \underline{\sigma}(I + F\Phi B), \quad \omega \geq 0$$

insures stability of the reduced order control.

3.2.3 Actuator/Sensor Dynamics

Frequency-domain robustness measures can be used to examine the effect of uncertainty in actuator/sensor dynamics. Consider the system of Figure 8, where H_P is the plant, H_A the actuator dynamics, H_S the sensor dynamics, and H_C the control. Suppose that H_A is perturbed by δH_A and (H_A, H_S) represent the nominal actuator/sensor dynamics. Breaking the loop at the actuator input gives,

$$G = H_C H_S H_P H_A (I + H_A^{-1} \delta H_A)$$

Identifying $H_A^{-1} \delta H_A$ as a multiplicative perturbation gives the perturbation bounds,

$$\bar{\sigma}(\delta H_A) < \underline{\sigma}(H_A)\, \underline{\sigma}(I + (H_C H_S H_P H_A)^{-1}), \quad \omega \geq 0$$

which insures robust stability, provided that $H_A^{-1} \delta H_A$ is stable, and H_C stabilizes the nominal system. A similar result can also be established for sensor perturbations.

This result has some interesting consequences. Suppose that for the nominal system, no other actuator/sensor or compensator gives better performance. In this case, the robustness test can evaluate candidate actuators which are not optimal. Tradeoffs in actuator parameters and robustness are easily established without the need for recalculating a new optimal control.

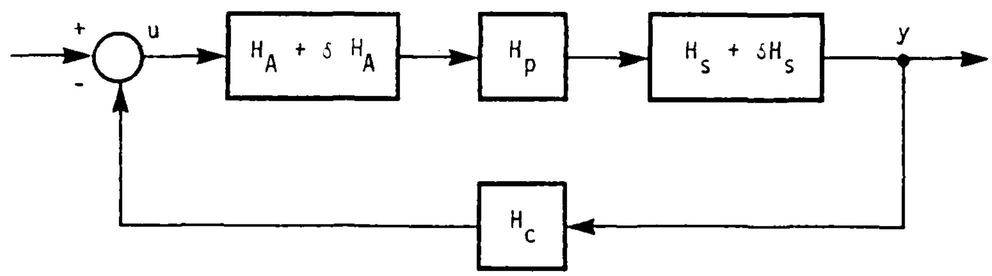

Figure 8 Actuator/Sensor Uncertainty

The effect of actuator/sensor parameter uncertainty and actuator/sensor failure can also be tested. If, uncertainty results from actuator/sensor failure the tests give an indication of reliability to failures in control/communication channels that are not catastrophic, i.e., do not destabilize.

It should be pointed out that where the loop is broken to calculate G is dependent on the uncertainty being tested. A common sense rule is to break the loop wherever nature is inserting a disturbance or uncertainty. There is no sense testing at a point where there is no effective disturbance.

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APPENDIX A: LINEAR PARAMETER INSENSITIVE CONTROLLERS AND FILTERS

Several methods are proposed for the control of systems whose model structure or model parameters are not known exactly. One class of control laws includes explicit real-time estimation of the system model and the control inputs may work explicitly towards enhancing future estimation accuracy (dual control). These control designs are usually complex requiring nonlinear controls, even for linear systems. This section presents a broad class of linear control laws which are minimally sensitive to variations in system parameters. It is shown that state-feedback controllers with constant gains are not optimal when the parameters of the system change. Parameter insensitive controllers use past states and controls for feedback and on periodic variations in control gains. These controllers do not estimate parameters or model structure and the system need not be excited to improve identifiability. They are usually simpler to design and implement than adaptive or dual controllers and often work better than these controllers if the parameters change relatively quickly. Classical control laws are shown to be a subset of these parameter insensitive controllers. The concepts and formulations of parameter insensitive controllers are also extended to parameter insensitive filters. Parameter insensitive compensators may be designed by combining a parameter insensitive filter and a parameter insensitive controller.

1.0 INTRODUCTION

Consider a linear system in which the nxl state variable x follows the difference equation

$$x(k+1) = Ax(k) + Bu(k) + w(k) \tag{1}$$

$u(k)$ is the qxl vector of controls and $w(k)$ is an nxl vector of zero mean which Gaussian state disturbance. A and B are state transition and control distribution matrices of appropriate dimensions. The design of control inputs for this system and its continuous time counter has been a problem of intense study. The control design problem consists of selecting the control input sequence $u(k)$ to satisfy certain constraints on states and controls and to optimize and objective function.

Most feedback control design studies have been limited to the assumption that A and B are known perfectly in the design stage. If the parameters of the system are known exactly and all states are measured with no noise, the control design problem simplifies considerably. In addition, for a criterion function which is a quadratic function of state and control variables

$$J = \sum_{k=1}^{N} y^T(k)Qy(k) \tag{2}$$

$$y(k) = Hx(k) + Du(k) \tag{3}$$

the optimal control input at any time is a linear function of state variables at that time. This control law has several desirable features including stability of the closed-loop system if (A,B) and $(A,Q^{\frac{1}{2}})$ are controllable.

In many problems of practical interest, all model parameters, and sometimes even the model order, are not known exactly. The state equation description matrices A and B may be considered functions of a vector of unknown parameters θ, whose values are not known exactly at the time of design but is known to lie in a set Ω. By selecting Ω appropriately this formulation may include cases where the model order is not known. The control design problem is much more complex in this case, even for quadratic performance indices. Past measurements of the state variable are then useful for controlling the system in the future, because past states contain information about system parameters. The usefulness of past data depends on how stationary parameter values are.

There are two ways of using past measurements to improve the behavior of the controlled system in the future: (a) the indirect approach, and (b) the direct approach. In the indirect approach, the past data is used to estimate the parameters of the system explicitly (see, e.g. [1], [2], [3]). In the direct approach, past measurements are used in the feedback control law without going through the intermediate parameter identifying step (see, e.g. [4], [5], [6]). When the indirect approach is used, the control laws may stimulate the system to improve future identification accuracy (active or dual control) or may be passive where learning is accidental (passive control).

The dual control law (indirect/active) is the optimal solution for controlling a system with partially known parameters. However, its computational complexity prevents real-time implementation even for very simple systems. The open-loop feedback optimal control law (indirect/passive) also requires a large computation time and is difficult to

implement for real systems. Because of the complexity of these schemes, global stability is often difficult to analyze. If the parameters change quickly, their estimates based on past data may be obsolete by the time enough data is collected to ensure reasonable estimation accuracy. In addition, it is not fruitful to spend control energy in improving the accuracy of estimates of changing parameters.

The direct approaches do not estimate the parameters of the system explicitly, instead the past measurements are used directly in a feedback control loop. If controllers based on a direct approach have fixed structure, parameter insensitive (rather than adaptive) control laws result. The performance of a parameter insensitive controller does not improve with time even if the parameters are constant, unlike an adaptive controller where average performance improves as more accurate estimates of parameters are obtained. Usually, however, parameter insensitive controllers are simpler to implement and analyze.

This paper deals with techniques for designing parameter insensitive control laws. Previous work in this area was done by Cruz, et al [4], Hadass and Powell [5], and more recently by Kleinman [6] and Padilla and Cruz [7]. All these techniques either modify the gains on the state variables resulting from the quadratic performance index or include quadratic functions of sensitivities of state variables to parameter variations in the performance index. The sensitivities of the performance measure to variations in parameters is usually decreased by increasing the feedback gains, thus speeding the dynamics. These methods do not use past measurements and may therefore only produce only a marginal reduction in parameter sensitivity in many cases.

Our approach for designing parameter insensitive controllers is significantly different from the previous approaches. We first show that control laws based on linear combinations of current state vector are, in general, nonoptimal with unknown parameters. The control laws are then extended (a) to include linear combinations of past states and inputs in addition to linear function of current state and (b) to include control gains which are periodic functions of time. Both of these controllers provide much better system performance than modern controllers. It is shown by using simple first-order linear system examples that systems for which stability cannot be assured over the range of parameter variations using conventional controllers (based on using linear combinations of states for control inputs) can be stabilized by using each of the controller structures proposed by us. Another important element of our parameter insensitive controllers is that they may be used for large as well as small variations in parameters. Similar results are derived for parameter insensitive filters also.

This paper is organized as follows. Section 2 describes two criteria of optimality for parameter insensitive controllers. Section 3 shows that the constant gain controllers are nonoptimal within the class of linear controllers when parameters are not known exactly. Sections 4 and 5 discuss two forms of parameter insensitive filters, while Section 6 extends the concepts to a parameter insensitive filter. Section 7 presents some important extensions, followed by an example in Section 8 and summary and conclusions in Section 9.

2.0 CRITERIA OF OPTIMALITY

Let us suppose that the control law is to be designed to minimize a performance index which is a function of output and control time histories.

Since the value of the performance index depends on unknown parameters for a controller with fixed structure, for large parameter variations we may define several criteria of optimality. Two of these are:

(1) Worst Case Design: In this case, the control law is selected to provide the minimum cost for the worst value of θ, i.e., the control law is selected such that

$$\max_{\theta \in \Omega} \min_{u} J(\theta, u)$$

This is also called a minimax design. The problem can be considered to be a two-person game in which the designer selects the control law and nature selects the parameter θ from the feasible set Ω.

(2) Best Average Design: The control law is selected to minimize the average value of the cost function over $\theta \epsilon \Omega$, i.e.,

$$\min_{u} \mathcal{E}_{\theta \epsilon \Omega} [J(\theta, u)]$$

where $\mathcal{E}$ is the expectation operator.

3.0 NONOPTIMALITY OF THE CONSTANT GAIN CONTROLLER

When the parameters of a linear system are known exactly, and the performance index is of the form of equations (2) and (3), the control input is a linear function of the state variables and the control gains reach steady state values as the time horizon is expanded. In this section, we show that, in general, linear constant gain controllers are nonoptimal from the class of linear controllers if the parameters may lie anywhere in

a specified region. For the sake of simplicity, we will prove this for a first-order continuous system with two possible sets of parameter values (extension of the results to higher order system with many possible parameter values is conceptually straightforward).

Consider a linear system with two possible descriptions

$$\begin{aligned} \dot{x}_1 &= a_1 x_1 + b_1 u + w \\ \dot{x}_2 &= a_2 x_2 + b_2 u + w \end{aligned} \tag{4}$$

Consider the performance index

$$J = \frac{1}{2T} \int_0^T (qx^2 + u^2)\, dt \tag{5}$$

and a control law of the form

$$u = cx$$

If the intensity of process noise is W and X_1 and X_2 are the covariances of the state under the two descriptions, follow the equations

$$\begin{aligned} \dot{X}_1 &= 2(a_1 + b_1 c) X_1 + W \\ \dot{X}_2 &= 2(a_2 + b_2 c) X_2 + W \end{aligned} \tag{6}$$

and

$$J = \frac{1}{2T} \int_0^T (q + c^2) X\, dt \tag{7}$$

The Hamiltonian for the best average design is

$$H = \sum_{i=1}^{2} \{\frac{1}{2} (q + c^2) X_i + \lambda_i (2(a_i + b_i c) X_i + W)\} \tag{8}$$

If $(a_1 + b_1 c)$ and $(a_2 + b_2 c)$ are both negative, a steady state of X is obtained as the solution of the following equations

$$X_1 = -\frac{W}{2(a_1 + b_1 c)} \qquad X_2 = -\frac{W}{2(a_2 + b_2 c)} \tag{9}$$

$$\lambda_1 = \frac{(q+c^2)}{2(a_1+b_1c)} \qquad \lambda_2 = -\frac{(q+c^2)}{2(a_2+b_2c)} \tag{10}$$

$$c = -\frac{(\lambda_1 b_1 X_1 + \lambda_2 b_2 X_2)}{(X_1+X_2)} \tag{11}$$

To check the optimality of the steady state solution, we check the second order necessary conditions by linearizing the state equations and the performance index about the steady state solution

$$\begin{pmatrix} \delta\dot{X}_1 \\ \delta\dot{X}_2 \end{pmatrix} = F \begin{pmatrix} \delta X_1 \\ \delta X_2 \end{pmatrix} + G\delta c$$

$$\delta^2 J = \frac{1}{T}\int_0^T (\delta X_1 \;\; \delta X_2 \;\; \delta c) \begin{pmatrix} 0 & H_{xc} \\ H_{xc}^T & H_{cc} \end{pmatrix} \begin{pmatrix} \delta X_1 \\ \delta X_2 \\ \delta c \end{pmatrix} dt \tag{12}$$

where

$$F = \begin{bmatrix} -2(a_1+b_1c) & 0 \\ 0 & -2(a_2+b_2c) \end{bmatrix}, \quad G = \begin{bmatrix} 2b_1X_1 \\ 2b_2X_2 \end{bmatrix}$$

$$H_{xc}^T = [c+b_1\lambda_1, \; c_2+b_2\lambda_2], \quad H_{cc} = X_1+X_2 \tag{13}$$

By performing a suitable set of transformations on the perturbation state vector, it is easily shown that [9] equations (12) may be written as

$$\delta\dot{Z} = (F-GH_{cc}^{-1}H_{xc}^T)\,\delta Z + G\delta c$$

$$\delta^2 J = \frac{1}{T}\int_0^T (\delta Z^T \delta c) \begin{pmatrix} -H_{xc}^T H_{cc}^{-1} H_{xc} & 0 \\ 0 & H_{cc} \end{pmatrix} \begin{pmatrix} \delta Z \\ \delta c \end{pmatrix} dt \tag{14}$$

To determine if $\delta^2 J$ can be negative, we use Willems [9] results on the positivity of the algebraic Riccati equation and follow a procedure similar

to the one used by Speyer [10] for showing the optimality of aircraft cruise. Willems shows that a necessary and sufficient condition for $\delta^2 J$ to be positive is that

$$Y(\omega) = H_{cc} - G^T(-j\omega - F^T + H_{cc}H_{cc}^{-1}G^T)^{-1} H_{xc}^T H_{cc}^{-1} H_{xc} (j\omega - F + GH_{cc}^{-1}H_{xc}^T)^{-1} G \tag{15}$$

be positive for all ω. For scalar H_{cc}, $Y(\omega)$ may be simplified to

$$Y(\omega) = H_{cc}\left[1 - \frac{\alpha\alpha^*}{(H_{cc}+\alpha)(H_{cc}+\alpha^*)}\right] \tag{16}$$

where

$$\alpha = H_{xc}^T (j\omega - F)^{-1} G \tag{17}$$

Clearly $Y(\omega)$ is negative for some ω iff

$$\left|\frac{\alpha}{H_{cc}+\alpha}\right| > 1 \tag{18}$$

or

$$\mathrm{Re}(\alpha) < -\frac{H_{cc}}{2} \tag{19}$$

where $\mathrm{Re}(\alpha)$ is the real part of α. It is easily shown that

$$\mathrm{Re}(\alpha) = \eta\Big[(a_1b_2 - a_2b_1)^2\{(a_1+b_1c)(a_2+b_2c) - \omega^2\}$$
$$+ 2\omega(a_1b_2 - a_2b_2 - a_2b_1)(b_1-b_2)(a_1+a_2+(b_1+b_2)c)\Big] \tag{20}$$

where η is a positive constant. Since (a_1+b_1c) and (a_2+b_2c) are negative, $\mathrm{Re}(\alpha)$ can be made as negative as desired by selecting a suitably large ω as long as $a_1b_2 \neq a_2b_1$. Therefore, $\delta^2 J$ can be made negative and the constant gain feedback solution is nonoptimal. It is interesting to note that if $a_1=a_2$ and $b_1=b_2$, $\mathrm{Re}(\alpha)=0$, and a constant gain controller is optimal.

Theorem: When the parameters of a linear system are not known exactly, a controller using constant gain feedback on the current state is in general nonoptimal in the class of linear controllers even in the steady state.

The proof for a first-order system has been given above. It can easily be extended for multivariable system (examining the system in modal form is one approach). We demonstrate using a simple example that a periodic controller may provide an improved overall performance than a constant gain controller.

Example: Periodic Control. Consider a first-order example

$$x(k+1) = ax(k) + u(k) + w(k) \tag{21}$$

where a is known to lie between 0 and 2. Clearly the controller $u = cx$ cannot stabilize the system for all values of a.

Consider a periodic control law with period 2 and gains c_1 and c_2. This gives

$$\begin{aligned} x(k+1) &= (a + c_1)x(k) + w(k) \\ x(k+2) &= (a + c_2)x(k+1) + w(k+1) \end{aligned} \tag{22}$$

The covariance of the state will alternate between

$$X_1 = \frac{q[1+(a+c_2)^2]}{1-(a+c_1)^2(a+c_2)^2}$$

$$X_2 = \frac{q[1+(a+c_1)^2]}{1-(a+c_1)^2(a+c_2)^2} \tag{23}$$

and stability requires

$$(a+c_1)(a+c_2) < 1 \tag{24}$$

Stability can easily be guaranteed, for example, by selecting $c_1 = -0.5$, $c_2 = -1.5$. If $c_1 = c_2$ (constant gain controller), the closed-loop system cannot be stabilized for all possible values of a.

Suppose the performance index is (x^2+u^2). The worst case design gives $c_1 = -0.37$, $c_2 = -1.83$ (the worst values of a are 0 or 2). Better design may be obtained by using a periodic control with a longer time period.

4.0 LINEAR OPTIMAL PERIODIC CONTROLLERS

Having shown in the previous section that constant gain linear controllers are nonoptimal for systems with unknown parameters, we investigate controllers with periodically varying gains. Consider the system with state equations (1), performance index of equations (2) and (3) and a control law of the form

$$u(k) = C_{k|m}\, x(k) \tag{25}$$

where $k|m$ is k modulus m and m is the period of the control law. The closed-loop system equations are

$$x(k+1) = (A+BC_{k|m})x(k) + w(k) \tag{26}$$

and the state covariance follows the equation

$$X(k+1) = (A+BC_{k|m})X(k)(A+BC_{k|m}) + W \tag{27}$$

$$X(k) = E(x(k)x^T(k)) \tag{28}$$

The periodic system is stable if and only if all eigenvalues of

$$\prod_{i=1}^{m} (A+BC_i) \tag{29}$$

are less than one. This condition shows in a qualitative way the reason for the effectiveness of the periodic controllers. If a constant gain controller is used the eigenvalues of $(A+BC)$ must be smaller than zero to ensure stability. The selection of a set of gain C_i in a period controller increases the flexibility in the choice of the feedback control structure to ensure stability in particular and to provide a more desirable performance in general. It is quite easy to see that the eigenvalues of the matrix of equation (29) may be made smaller than one for much larger variation in elements of A and B than the eigenvalues of the matrix $(A+BC)$. This result agrees with the theorem proved in the previous section.

Having selected the periodicity m of the controller the optimal periodic feedback gains may be computed as follows. Even though a time varying periodic gain may be computed we will only consider the steady state case. If the closed-loop system is stable, i.e. all eigenvalues of the matrix of equation (29) are smaller than one, the periodic steady state covariances follow the equation

$$X_{i+1} = (A+BC_i)X_i\,(A+BC_i)^T + W \qquad i = 1,2,\ldots,m-1$$
$$X_1 = (A+BC_m)X_m\,(A+BC_m)^T + W \tag{30}$$

The average steady-state cost function over the period of the control law is

$$J = \mathrm{Tr}\,\frac{1}{m}\sum_{i=1}^{m} (H+DC_i)^T Q\,(H+DC_i)X_i \tag{31}$$

The periodic controller may be looked upon as a steady-state controller with increased number of state variables. The control gains may be determined for the best average design or the worst case design as follows.

Best Average Design. The cost function for the best average design is

$$J = \underset{\theta\varepsilon\Omega}{E}\ \mathrm{Tr}\,\frac{1}{m}\sum_{i=1}^{m} (H+DC_i)^T Q\,(H+DC_i)X_i \tag{32}$$

Adjoining the constraints of equation (30), we get

$$J = \underset{\theta\varepsilon\Omega}{E}\ \mathrm{Tr}\,\frac{1}{m}\sum_{i=1}^{m} (H+DC_i)^T Q\,(H+DC_i)X_i$$
$$+ \sum_{i=1}^{m} \Lambda_i\,\{(A+BC_i)X_i(A+BC_i)^T + W - X_{i+1}\} \tag{33}$$
$$X_{m+1} = X_1$$

The optimality conditions are

$$\Lambda_{i-1} = (A+BC_i)^T\Lambda_i(A+BC_i) + \frac{1}{m}(H+DC_i)^T Q(H+DC_i) \qquad i=1,\ldots,m$$
$$\Lambda_o = \Lambda_m \tag{34}$$

and

$$\underset{\theta\varepsilon\Omega}{E}\ \{\frac{1}{m}X_i(H+DC_i)^T QD + X_i(A+BC_i)^T\Lambda_i G\} = 0 \qquad i=1,2,\ldots,m \tag{35}$$

The optimal gains C_i are obtained by solving linear equations (30) and (34) for each possible value of θ and solving linear equations (35). An iterative procedure is required to obtain a solution to these equations.

Worst Case Design. The worst case controller is obtained by solving equations (30) and (34) and selecting θ and C according to the following expression

$$\max_{\theta\epsilon} \min_{C_1,C_2,\ldots,C_m} \left\{ \frac{1}{m} \sum_{i=1}^{m} (H+DC_i)^T Q (H+DC_i) X_i \right\}$$

$$+ \sum_{i=1}^{m} \Lambda_i \{ (A+B_i C) X_i (A+B_i C)^T + W - X_{i+1} \} \tag{36}$$

Therefore, we must seek a saddle point solution to solve for θ and C_i, $i=1,2,\ldots,m$. The gains must be constrained such that the closed-loop system is stable.

In the next section, we analyze another new type of linear controller structure which decreases parameter sensitivity of the closed-loop system.

5.0 CONTROLS LINEAR FUNCTIONS OF PRESENT AND PAST STATE AND INPUT VARIABLES

When the parameters of a system are known exactly, the present value of the state vector contains all the information about the history of the system. If the parameters are not known, however, the past measurements may contain significant information about the nature of the system. All adaptive schemes make use of past measurements to obtain an improved control input. Nonlinear functions of measurements are used to estimate parameters in the indirect approaches and are used directly in other adaptive control schemes. When the parameters are estimated they are used in feedback control calculations. Even for linear open-loop system dynamics, these approaches lead to complex nonlinear closed-loop systems. As mentioned in Section 1, even the stability of such systems may be difficult to ascertain under all possible conditions.

Because past mesaurements contain information about system parameters, they may be used in parameter insensitive control schemes. It is simplest

to use past measurements directly in a linear feedback controller. This scheme extracts a large portion of total information available from the past measurements and provides a simple control law whose properties are easy to analyze and whose stability regions may be easily derived.

We again consider a system and a performance index of equations (1)-(3) and a control input which is a linear function of past states

$$u(k) = \sum_{i=1}^{m} C_i x(k-i+1) \tag{37}$$

Then the closed-loop system follows the equations

$$x(k+1) = (A+BC_1)x(k) + \sum_{i=2}^{m} BC_i x(k-i+1) \tag{38}$$

If we define

$$A' = \begin{bmatrix} A+BC_1 & BC_2 & BC_3 & \cdots & BC_m \\ I & 0 & 0 & & 0 \\ \vdots & I & \vdots & \ddots & \\ \vdots & \vdots & \vdots & & 0 \\ 0 & 0 & 0 & \cdots\ I & 0 \end{bmatrix} \tag{39}$$

$$W' = \begin{bmatrix} W & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \\ 0 & 0 & 0 & & 0 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \tag{40}$$

$$x'(k) = \begin{bmatrix} x(k) \\ x(k-1) \\ \vdots \\ x(k-m+1) \end{bmatrix} \tag{41}$$

and

$$X'(k) = E[x'(k)\ x'^T(k)] \tag{42}$$

Then

$$X'(k+1) = A'X'(k)\ A'^T + W' \tag{43}$$

The gains C_i's may be selected as time varying functions or as constants to optimize the performance index of equation (2). To keep control laws simple we will discuss the case where C_i's are constant, nonperiodic functions.

In terms of $X'(k)$, the performance index of equations (2) and (3) is written as

$$J = \mathrm{Tr} \sum_{k=1}^{N} (H'+D'C')^T Q'(H'+D'C')X' \tag{44}$$

where

$$H' = \begin{bmatrix} H & 0 & \cdots\cdots & 0 \\ 0 & 0 & & \vdots \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots\cdots & 0 \end{bmatrix} \tag{45}$$

$$D' = \begin{bmatrix} D \\ D \\ \vdots \\ D \end{bmatrix} , \quad C' = [C_1 \quad C_2 \quad \ldots \quad C_m] \tag{46}$$

and

$$Q' = \begin{bmatrix} Q & 0 & \cdots\cdots & 0 \\ 0 & 0 & \cdots\cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots\cdots & 0 \end{bmatrix} \tag{47}$$

The constant C_i's are based on steady state system operation. A steady state solution to equation (44) is obtained if and only if all poles of the matrix A', defined by equation (45) are within the unit circle. The steady state value of X' is then

$$X' = A'X'A'^T + W' \tag{48}$$

and the performance index becomes

$$J = \mathrm{Tr}\ (H'+D'C')^T Q'(H'+ D'C')X' \tag{49}$$

<u>Worst Case Design</u>. In the worst case design the performance index becomes

$$J = \max_{\theta\varepsilon\Omega} \mathrm{Tr}(H'+D'C')^T Q'(H'+D'C')X' \tag{50}$$

Adjoining the constraint of equation (48) using Lagrange matrix Λ', it is easily shown that

$$\Lambda' = A'^T\Lambda'A' + (H'+D'C')^T Q'(H'+D'C') \tag{51}$$

and the gain C' is obtained from

$$\max_{\theta\varepsilon\Omega}\ \min_{C'} \mathrm{Tr}[(H'+D'C')^T Q'(H'+D'C')X'+ \Lambda'(A'X'A'^T+W'-X')] \tag{52}$$

<u>Best Average Design</u>. The performance index for best average design is

$$J = \mathop{E}_{\theta} \mathrm{Tr}(H'+D'C')^T Q'(H'+D'C')X' \tag{53}$$

Again using the Labrange matrix we must use equation (51) together with

$$\mathop{E}_{\theta} [X'(H'+D'C')^T Q'D' + X'A'^T\Lambda'B'] = 0 \tag{54}$$

$$B' = \begin{bmatrix} B \\ B \\ \vdots \\ B \end{bmatrix} \tag{55}$$

Best Average Design for Small Parameter Variations. Let us suppose that there is a single parameter ϑ which varies in a small region about the nominal value ϑ_o. Then equation (54) may be written as

$$\underset{\vartheta}{E}\left[\xi(\theta_o) + \frac{\partial\xi}{\partial\theta}\Delta\theta + \frac{\partial^2\xi}{\partial\theta^2}\Delta\theta^2 \; \ldots\ldots \right] = 0 \tag{56}$$

$$\xi = X'(H'+D'C')Q'D' + X'A'^{-T}\Lambda'B' \tag{57}$$

If the parameter does not change too much, we can neglect all terms of order higher than two. Let V be the variance of ϑ. Then equation (56) is approximated as

$$\xi(\theta_o) + \frac{\partial^2\xi}{\partial\theta^2} V = 0 \tag{58}$$

$\frac{\partial^2\xi}{\partial\theta^2}$ is computed by differentiating appropriate quantities and is related to the information matrix of parameters ϑ. Thus the approach of Kleinman and Rao [12] is a special case of the proposed approach.

Selection of m. The selection of m is based on the conflicting requirement of a simple control law and reduced storage requirements against a smaller performance index. The marginal improvement often decreases as m is increased. In practice, it is best to start with a small m and then to increase it if the overall performance is unsatisfactory.

Special Case. As mentioned before, all previous formulations for parameter insensitive controllers are based on using only a linear function of the current state in feedback. Our formulation of this section may be specialized to previous approaches by removing the superscript prime from variables in each of the equations. It is obvious, however, that use of past state variables will lead to a lower performance index.

6.0 LINEAR PARAMETER INSENSITIVE FILTER

The ideas presented in the previous section to design a parameter insensitive controller may be extended to parameter insensitive filters. An optimal filter for the system of equations (1) and (3) is given by

$$\hat{x}(k+1|k) = A_f\hat{x}(k|k) + B_f u(k)$$

$$\hat{x}(k|k) = \hat{x}(k|k-1) + K\nu(k)$$

$$\nu(k) = y(k) - H_f\hat{x}(k|k-1) - D_f u(k) \tag{59}$$

where $x(k|\ell)$ is the estimate of $x(k)$ based on measurements till time ℓ. The gain K is selected by solving a Riccati equation [13]. The subscript f represents the value of the quantity used in the filter. Clearly,

$$\tilde{x}(k+1|k) = A_f(I-KH_f)\tilde{x}(k|k-1) + (\tilde{A}-A_fK\tilde{H})x(k) + (\tilde{B}-A_fK\tilde{D})u(k) + w(k) - K\nu(k) \tag{60}$$

where

$$\tilde{x}(k|k-1) = x(k) - \hat{x}(k|k-1)$$
$$\tilde{A} = A-A_f \tag{61}$$

and $w(k)$ and $v(k)$ are processes and measurement noise sources with covariances W and V, respectively. If A, B, H, and D are known accurately, second and third terms on the right-hand side of equation (6) drop out and the gain K computed from the Riccati equation gives the optimal state estimate. When the parameters of a system may lie anywhere in a region, the estimation error is likely to be much higher than normal if the gain K is based on a particular Riccati equation solution. In fact, the error covariance may be larger than the state covariance (indicating a zero estimate for the state vector is better than the filtered estimate).

There are several parameter insensitive filter formulations, which may be used to reduce sensitivity to parameter variations. Three approaches, analogous to those used for parameter insensitive controller designs, are presented here.

A. Direct Computation of Gain

The gain K (and possibly A_f, B_f, H_f and D_f) may be selected by direct optimization to provide the best compromise of estimation errors over the range of parameter variations. The state and the state estimation error follow the equations

$$x'(k+1) = A'x'(k) + B'u(k) + \Gamma'w'(k) \tag{62}$$

$$x'(k) = \begin{bmatrix} x(k) \\ \tilde{x}(k|k-1) \end{bmatrix}$$

$$A' = \begin{bmatrix} A & 0 \\ (\tilde{A}-A_f\tilde{K}\tilde{H}) & A_f(I-KH_f) \end{bmatrix}, \quad B' = \begin{bmatrix} B \\ \tilde{B}-A_f\tilde{K}\tilde{D} \end{bmatrix}, \quad \Gamma' = \begin{bmatrix} I & 0 \\ I & -K \end{bmatrix} \tag{63}$$

Both the state and its estimation errors have deterministic and stochastic components.

$$x_D'(k+1) = A'x_D'(k) + B'u(k) \tag{64}$$

$$x_S'(k+1) = A'x_S'(k) + \Gamma'w'(k) \tag{65}$$

Let

$$X_S' = \mathcal{E}[x_S'(k+1)\; x_S'^T(k+1)] \quad \text{and} \quad W' = \begin{bmatrix} W & 0 \\ 0 & V \end{bmatrix}$$

$$X_S'(k+1) = A'X_S'(k)A'^T + \Gamma'W'\Gamma'^T \tag{66}$$

If the mean square error is to be minimized in the best average sense, the following problem must be solved

[illegible]

Note that [illegible] the [illegible] depend on gain [illegible] optimization [illegible] B_i, H_i and [illegible]. The formulation [illegible] conceptually similar if the filter is stable; the filter [illegible] selected for best performance in steady state.

B. Periodic Gain

The filter gain [illegible] may be selected [illegible] period of the control law is [illegible] the filter equations [illegible]

$$\hat{x}(k+1|k) = A_i \hat{x}(k|k) + B_i u(k)$$

$$\hat{x}(k|k) = \hat{x}(k|k-1) + K_{k \bmod m}\, \nu(k)$$

$$\nu(k) = y(k) - H_i \hat{x}(k|k-1) - [illegible]$$

The estimation error is again given by equations [illegible] except that K is a function of time. The optimization procedure is similar to equation (67) and the gains could be based on periodic steady state performance.

C. Feedback on Past Innovations

When the filter is nonoptimal, the innovations are nonwhite. Past innovations may then be used to improve the estimate of the current state. The filter would then follow equations [illegible] and

$$\hat{x}(k|k) = \hat{x}(k|k-1) + \sum_{i=1}^{m} K_i\, \nu(k-i+1)$$

The estimation error covariance equations are more complex now. The optimal set of filter gains may, however, be selected as for the controller using past states for feedback.

[illegible]

[illegible] The [illegible] law structure of [illegible] include past inputs, which also contain information [illegible]. A more general and superior formulation for para- [illegible]

$$[illegible] + \sum_{i=1}^{m} [illegible]\, k-i) + [illegible]\, k-i) \quad (72)$$

[illegible] includes both the past states [illegible], the control law may be determined both for [illegible] designs.

[illegible] general formulation of equation (72), the control law may be [illegible] selecting a transfer function between the control [illegible] states. The transfer function provides an optimal [illegible] for all feasible parameter values. The [illegible] the closed-loop system will improve as the order of [illegible] increased, at the cost of increased complexity. [illegible] shows the relationship between the [illegible] and parameter insensitive controllers. Our analysis [illegible] controllers possess limited parameter in- [illegible] implicitly extended the concepts of [illegible] parameter variation margins. The special case [illegible] corresponds to a transfer function with zeroes but no [illegible]

Selection of control laws of the forms of equations (37) and (72) adds new poles to the system and may change the location of closed-loop system zeroes. The ability to move the zeroes of the closed-loop system is partly responsible for the increased capability to minimize effects of parameter variations.

B. <u>A Reformulation</u>: The control law of equation (72) may be reformulated as

$$u(k) = \sum_{i=1}^{m} C_i x^{\star}(k-i+1) \tag{73}$$

where

$$x^{\star}(k+1) = A_{\varepsilon}^{\star} x^{\star}(k) + B_{\varepsilon}^{\star} u(k) + K_{\varepsilon}^{\star} x(k) \tag{74}$$

leading to the transfer function

$$u = (I + \sum_{i=1}^{m} C_i Z^{i-1}(Z^{-1}I-A^{\star})^{-1}B^{\star})^{-1} \sum_{i=1}^{m} C_i Z^{i-1}(Z^{-1}I-A^{\star})^{-1}K^{\star}x \tag{75}$$

where

$$Z^{-1}x(k) = x(k+1).$$

Equation (72) leads to the following transfer function

$$u = \left[I + \sum_{i=1}^{m} G_i Z^i \right]^{-1} \left[C_o + \sum_{i=1}^{m} C_i Z^i \right] x \tag{76}$$

Clearly the two formulations may be made equivalent.

$A_{\varepsilon}^{\star}$, $B_{\varepsilon}^{\star}$, and $K_{\varepsilon}^{\star}$ are selected in a suitable manner to optimize the performance index. <u>Note $x^{\star}(k)$ may be of lower, higher or the same order as the state vector</u>. The approach proposed by Padilla and Cruz [7] is a special case of the general formulation presented here. Padilla and Cruz try to design optimal filter/controller combinations to minimize sensitivity

to small parameter variations. Our approach also brings out the philosophy and the reasons for using the suggested formulation.

C. Continuous Systems: For a system in differential equation form, the following form may be selected for parameter insensitive controllers instead of the form of equation (72).

$$u^m(t) = \sum_{i=0}^{m-1} C_i x^i(t) + \sum_{i=0}^{m_2} D_i u^i(t) \tag{77}$$

where the superscript denotes the order of differentiation.

In order to avoid differentiating signals $m-1 \geq m_2$. C_i and D_i could be selected using procedures similar to those for discrete systems. A period control law with time period T may be selected as follows.

$$u(t) = C(t-mT)x(t), \quad mT < t \leq (m+1)T \tag{78}$$

D. Design of Parameter Insensitive Compensator: Parameter insensitive controllers of Sections 4, 5 and 6 may be designed together with parameter insensitive filters of Section 7 to obtain a parameter insensitive compensator. A direct approach may also be used in which we determine an optimal transfer function between the control input and the measured output. Note that the Kalman filter used in a feedback controller is a special case of this general formulation.

8.0 EXAMPLE

The example in this section is about a simple first-order system demonstrating how use of past outputs in a linear fashion can tremendously decrease parameter sensitivity of the closed-loop system.

Example: We take the same system and show the use of past feedback to achieve insensitivity

$$x(k+1) = ax(k) + bu(k) + w(k) \tag{75}$$

Case 1: b known exactly and assume it equals one. Let

$$u(k) = c_1x(k) + c_2x(k-1) \tag{76}$$

The region of stability for various choices of c_1 and c_2 is shown in Fig. 1. For nonzero c_2 and requirement of closed-loop stability, the maximum allowable variation in parameter a doubles as compared to zero c_2 (conventional controller). Also shown in the figure are curves of constant mean square state excursion, X. If $W_1=1$ and $W_2=0$, and parameter a varies within $a_o \pm \Delta a$, optimal $c_1=-a_o$ and optimal c_2 is shown in Fig. 2 for best average and worst case designs. The sensitivity of the response may be further reduced by incorporating $x(k-2)$ in control $u(k)$.

Case 2: b not known exactly. This controller may be used to decrease sensitivity to variation in b also. Fig. 3 shows maximum allowable variation in control gain b as a function of a_o and Δa. The effect of not using $x(k-1)$ in the feedback is to reduce allowable variation in b (see Fig. 3).

9.0 SUMMARY AND CONCLUSIONS

This paper discusses passive controller structures which minimize sensitivities of closed-loop system to variations in system parameters. These controllers do not possess any learning properties, therefore, their performance does not improve with time. However, they are simple to implement, and their stability regions may be determined using well-known techniques.

We developed two linear parameter insensitive controllers, one based on a periodic function of state variables and the other on a linear

combination of past states and inputs. Using past state and inputs in the control law is consistent with adaptive controllers, which are based on nonlinear functions of past variables. Leads and lags used in classical control laws are also special cases of the parameter insensitive controllers suggested by us in the paper.

The concepts used in formulating parameter insensitive controllers are extended to the design of parameter insensitive filters. When the parameters are known exactly and an optimal filter is used, the innovations are white [11], but with unknown parameters the innovations are in general correlated. Past innovations are therefore used in addition to the current innovations to update the state estimate at each measurement. A periodically varying filter gain matrix may also be used to minimize the effect of parameter variations.

In feedback control problems with noisy measurements, a parameter insensitive filter may be combined in series with a parameter insensitive controller to give a minimally sensitive compensator. The performance index for the overall system may sometimes be optimized directly to give a compensator using past as well as current measurements and inputs for feedback.

Parameter insensitive controllers and filters presented in this paper extend the structure of controllers for completely known systems to partially known system logical manner. Our formulation provides a natural bridge between classical and modern controllers as well. Classical controllers based on incorporating appropriate leads and lags are often significantly less sensitive than modern controllers, but do not possess the same optimality properties. All classical controllers are subsets of our linear parameter

insensitive controllers and optimality properties can easily be imposed using our approach. This fact may explain some of the well known properties of classical controllers.

The advantages of our controllers are: (a) they are simple to design and implement, (b) a desired tradeoff may be made between optimality and complexity, (c) the stability properties and stability regions are easily determined, (d) the structure of the controller does not change as long as the parameters vary within the prescribed set, and (e) the controller works well when parameters change relatively fast. A disadvantage is that there is no learning, which may be important when parameters remain constant over long periods. This is often a minor restriction, however, because parameter insensitive controllers are required most when parameters do in fact change with time.

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FIGURES

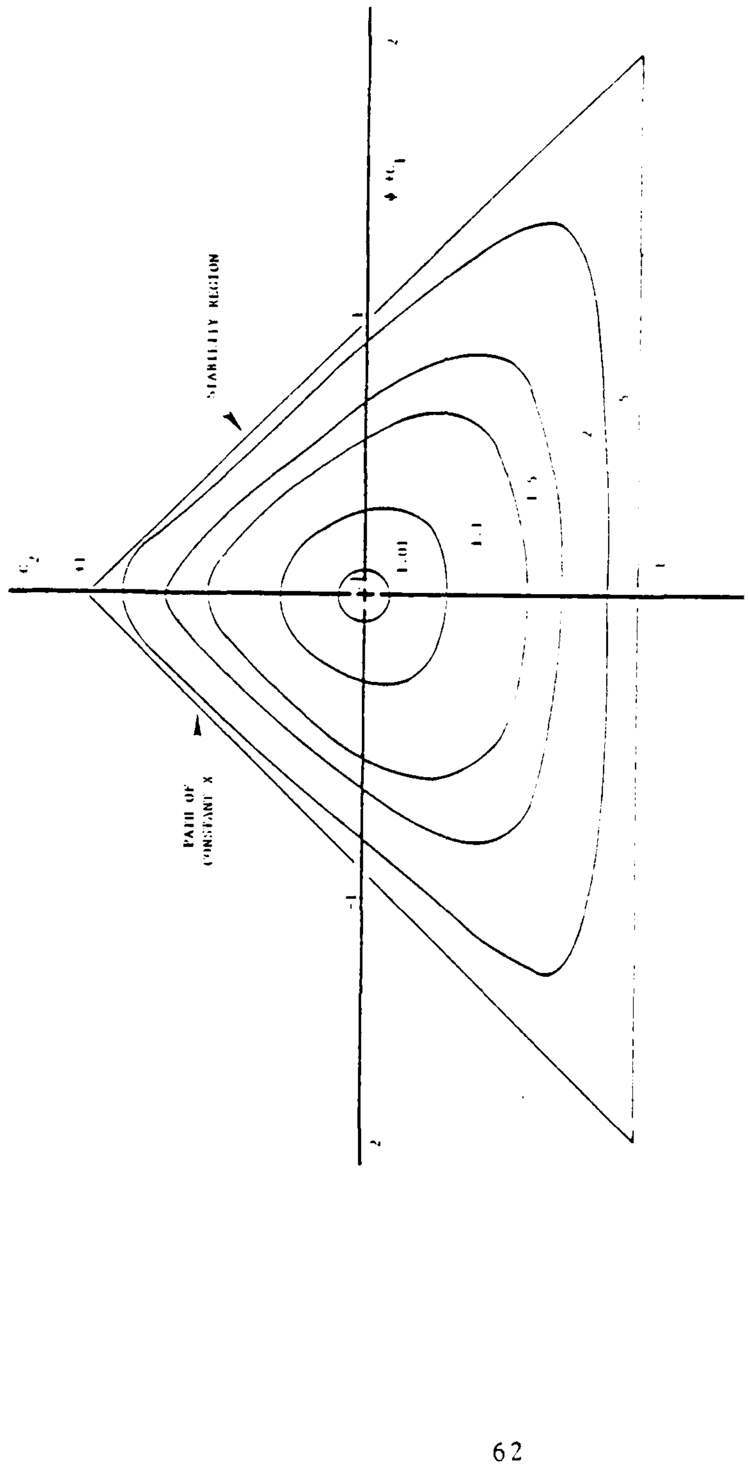

Figure 1

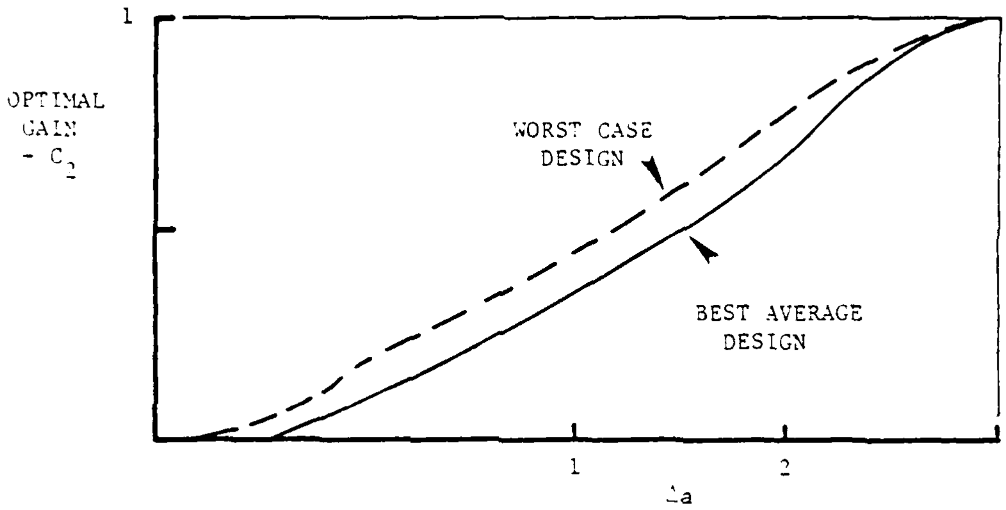

Figure 2

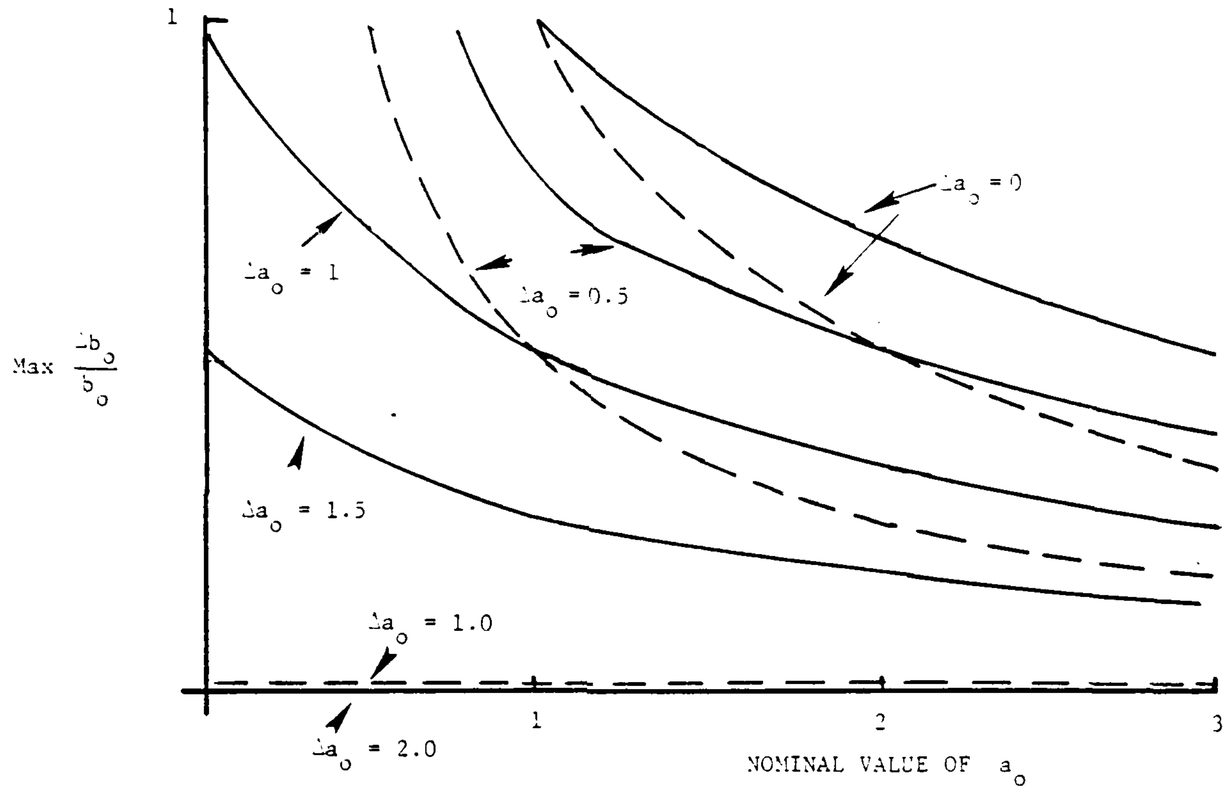
1
$\Delta a_o = 0$
$\Delta a_o = 1$
$\Delta a_o = 0.5$
Max $\frac{\Delta b_o}{b_o}$
$\Delta a_o = 1.5$
$\Delta a_o = 1.0$
1
2
3
$\Delta a_o = 2.0$
NOMINAL VALUE OF a_o

Figure 3

APPENDIX B: FREQUENCY-SHAPED COST FUNCTIONALS: AN EXTENSION OF LQG DESIGN METHODS

The linear-quadratic-Gaussian (LQG) method for feedback control design is extended to include frequency shaped weighting matrices in the quadratic cost functional. This extension provides a means to meet classical design requirements with automated computational procedures of modern control theory. A design algorithm to optimize frequency shaped cost functionals requires definition of new states and the solution of a modified LQG problem. Four examples are presented to demonstrate frequency shaping methodology: (1) aircraft in lateral wind, (2) an industrial crane, (3) vibration control in helicopters, and (4) a controller based on a reduced order dynamic model.

I. INTRODUCTION

Modern control theory method of linear-quadratic-gaussian (LQG) design is based on linear model, Gaussian noise and quadratic performance index. The state variable model is

$$\dot{x} = Fx + Gu + w, \qquad x(0) = x_o, \quad 0 \leq t \leq t_f \tag{1}$$

where x is nx1 state vector, u is qx1 control vector and w is nx1 Gaussian noise with mean zero and intensity Q. The control u is selected to minimize:

$$J = \int_0^{t_f} (x^T A x + u^T B u)dt + x^T(t_f) S_f x(t_f) \tag{2}$$

A is a positive semi-definite matrix and B is a positive definite matrix. The optimal control law is linear[1].

$$u^*(t) = C(t)x(t) \tag{3}$$

C(t) is obtained explicitly by solving the following Riccati equation

$$\dot{S} = F^T S - SF - A + SGB^{-1}G^T S, \quad S(t_f) = S_f \tag{4}$$

$$C(t) = -B^{-1}G^T S \tag{5}$$

This equation must be solved backward in time. The control law is dependent on system definition matrices F and G and performance index weighting matrices A, B and S_f. The control gain C(t) is a function of time.

As $T \to \infty$, the solution to Eq. (4) reaches a steady state. The steady-state solution satisfies the algebraic Riccati equation (ARE)

$$-F^TS - SF - A + SGB^{-1}G^TS = 0 \qquad (6)$$

The feedback control signal requires all system states which, in general, are not measured. The states may be estimated by using an observer with measurements $y(t)$, which are contaminated with noise v

$$y = Hx + v \qquad (7)$$

$$\dot{\hat{x}} = F\hat{x} + Gu + K(y - H\hat{x}) \qquad (8)$$

In equation (8), $\hat{x}$ is an estimate of the state vector and K is selected to minimize estimation error based on the process and the measurement noise. The control law is modified to use the estimated state vector instead of the real state.

$$u = C\hat{x} \qquad (9)$$

It is clear from the above that the modern control theory is based on optimization methods with a specified model form. The feedback system behaves well under the following conditions.

(1) The model is valid for all values of inputs and states. In addition, the dynamics is well described at all input and state frequencies.

(2) The filter design also assumes that the dynamics is known equally accurately at all frequencies (The filter uses values of F, G and H matrices explicitly in addition to the gain K which also depends on the state definition matrices). This may make the combination of the filter and the control law extremely sensitive to errors, see also Doyle[2].

(3) The optimality of the filter is strongly dependent on the accuracy of noise statistics.

<u>Classical control design</u> methods are based on frequency domain descriptions of systems and often account for model uncertainty at high frequencies by the use of gain and phase margins that are larger than those needed if the model were certain. The system dynamics may be described in terms of the output-input transfer function (s is the Laplace variable)

$$y(s) = T(s)\, u(s) \tag{10}$$

Bode plots, Nyquist diagrams or other frequency domain methods are used to develop a feedback compensator. In general, the control law is of the form

$$u(s) = C_c(s)y(s) \tag{11}$$

For single-input, single-output systems, the compensator $C_c(s)$ is obtained through the use of root-locus, Bode or Nyquist plots. The solution procedure is complex for multi-input, multi-output systems. Rosenbrock[3], McFarlane[4], and others have done much work in this area, though the practical application of these techniques is still quite difficult.

II. PROPERTIES OF LQG AND CLASSICAL CONTROLLERS

The LQG controllers have a fixed structure described previously. The closed loop transfer function is of the form (ω is the frequency)

$$x(j\omega) = (j\omega I - F - GC)^{-1} w(j\omega) \tag{12}$$
$$y(j\omega) = H(j\omega I - F - GC)^{-1} w(j\omega)$$

It has been shown by Anderson and Moore[5] and Athans and Safonov[6] that the controller has a 60° phase margin and 50% to infinite gain margin. The phase and gain margin properties provide for a constant phase error in all channels and for individual variations in gain. The system may be extremely sensitive if two gains change in opposite directions. In addition, the phase and gain margin property does not relate directly to parameter sensitivity. When a filter is used for state estimation, the gain and phrase margin properties are no longer valid. In addition, the filter dynamics, dictated by specifie noise characteristics, may be too fast, leading to interaction with unmodeled terms. A very definite attention must be given to the problem of filter design when estimated states are used in LQG controllers. The transfer functions of the closed loop system when a filter is used for state estimation is given by

$$x(j\omega) = [j\omega I-F-GC(j\omega I-F-GC+KH)^{-1}KH]^{-1}w(j\omega) \tag{13}$$
$$y(j\omega) = Hx(j\omega)$$

Note that F+GC-KH may have poles in the right half-plane, leading to right half-plane closed-loop zeros, which sometimes cause additional sensitivity.

The closed-loop transfer function has n poles and, at most n-1 zeros. Therefore, the frequency response may drop off as slowly as $\frac{1}{\omega}$ at high frequencies where the model is not valid

(higher frequency modes are often neglected). This kind of response may be undesirable in many closed-loop systems. In addition, the controller has a tendancy to push all responses to higher frequencies resulting in more stringent actuator and sensor constraints. These LQG properties are a consequence of the optimization problem which assumes the model is valid at all frequencies and for all deviations of the state variable.

The classical controllers when properly designed, work well. The general form of the compensator matrix $C_c(s)$ provides enough flexibility for a variety of control design requirements. The major problem is the difficulty in obtaining the control law. The closed loop transfer function between the output and the noise is

$$y(j\omega) = [I + T(j\omega)C_c(j\omega)]^{-1}T_w(j\omega)w(j\omega)$$

where $T_w(j\omega)$ is the noise to output transfer function.

LQG methods with frequency-shaped cost functionals are developed to incorporate good classical control properties in modern control design methods, such that, automated computational procedures may be used to compute control laws which now require more difficult graphical approaches.

The next two sections present the theoretical background and a design algorithm for the frequency-shaping approach.

III. FREQUENCY-SHAPING OF COST FUNCTIONALS

To understand the concept of frequency shaping, it is necessary to write the standard LQG cost functional of Eq. (2) in the frequency domain. With infinite time horizon and no weighting on the final state, the cost functional may be written in the frequency domain using Parseval's theorem

$$J = 1/2 \int_{\infty}^{\infty}]x^*(j\omega)Ax(j\omega) + u^*(j\omega)Bu(j\omega)[d\omega \qquad (15)$$

where * implies complex conjugate. Clearly, in this formulation, the weighting matrices are not functions of frequency, i.e., the state and control excursions at all frequencies are considered equally unacceptable. In many systems, on the other hand, inputs in the neighborhood of a particular frequency are not desirable because of poor sensor, actuator, or model characteristics at that frequency. Historically, this constraint of constant weighting at all frequencies has resulted because of the difficulty of shaping the weighting functional with frequency in the conventional time domain LQG formulation. Representation of the cost functional in frequency domain provides a clue to the use of frequency shaping ideas in modern control theory techniques. Matrices A and B in Eq. (15) may be made functions of frequency to give a generalized cost functional

$$J = \tfrac{1}{2} \int_{\infty}^{\infty} [x^*(j\omega)A(j\omega)x(j\omega) + u^*(j\omega)B(j\omega)\, u(j\omega)]\, d\omega \qquad (16)$$

$A(j\omega)$ and $B(j\omega)$ are Hermitian matrices at all frequencies. It appears that a solution can be guaranteed if $B(j\omega)$ is positive definite and $A(j\omega)$ is positive semi-definite at all but a finite number of discrete frequencies (though this is not a necessary condition). It should be pointed out that even under these constraints, the solution may not be easy to find and in fact may not even be causal. The total class of weighting functions for which a causal solution may be found will be subjects of future research.

Polynomial weighting matrices A and B are studied in the following section.

IV. CONTROL LAW DESIGN

If the weighting functions $A(j\omega)$ and $B(j\omega)$ are assumed to be rational functions of squared frequency, ω^2, a systematic control design procedure, may be developed for positive semi-definite $A(j\omega)$ and positive definite $B(j\omega)$. This is not a serious limitation because a wide variety of functional forms may be approximated by ratios of polynominals. To develop a specific control design procedure, it is further assumed that $A(j\omega)$ has rank p, and $B(j\omega)$ is positive definite with full rank, q

$$A(j\omega) = P_1^*(j\omega)P_1(j\omega) \tag{17}$$

$$B(j\omega) = P_2^*(j\omega)P_2(j\omega) \tag{18}$$

P_1 and P_2 are pxn and qxq rational matrices. Define

$$P_1(j\omega)x = x^1 \tag{19}$$

$$P_2(j\omega)u = u^1 \tag{20}$$

If $P_1(j\omega)$ is a ratio of polynominals in $j\omega$ and the number of zeros does not exceed the number of poles, eq. (19) may be written as a system of differential equations with output x^1.

$$\begin{aligned} \dot{z}_1 &= F_1 z_1 + G_1 x \\ x^1 &= H_1 z_1 + D_1 x \end{aligned} \tag{21}$$

D_1 is zero if the number of poles is at least one more than the number of zeros. Equation (20) may also be written in terms of a differential equation, again if the number of zeros does not exceed the number of poles.

$$\begin{aligned} \dot{z}_2 &= F_2 z_2 + G_2 u \\ u^1 &= H_2 z_2 + D_2 u \end{aligned} \tag{22}$$

The dynamic Eq. (1) and the cost functional (16) may now be written in terms of an extended state vector.

$$\frac{d}{dt}\begin{bmatrix} x \\ z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} F & 0 & 0 \\ G_1 & F_1 & 0 \\ 0 & 0 & F_2 \end{bmatrix}\begin{bmatrix} x \\ z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} G \\ 0 \\ G_2 \end{bmatrix} u \tag{23}$$

$$J_{ss} = E(x^T \; z_1^T \; z_2^T \; u^T)\begin{bmatrix} D_1^T D_1 & D_1^T H_1 & 0 & 0 \\ H_1^T D_1 & H_1^T H_1 & 0 & 0 \\ 0 & 0 & H_2^T H_2 & H_2^T D_2 \\ 0 & 0 & D_2^T H_2 & D_2^T D_2 \end{bmatrix}\begin{bmatrix} x \\ z_1 \\ z_2 \\ u \end{bmatrix} \tag{24}$$

Defining appropriate vectors and matrices, Eqs. (23) and (24) become

$$\dot{X} = F^1 X + G^1 u \tag{25}$$

$$J = \lim_{T\to\infty} \frac{1}{2T}\int_0^T \left| X^T u^T \right| \begin{vmatrix} A & N \\ N^T & B \end{vmatrix} \begin{vmatrix} X \\ u \end{vmatrix} dt \tag{26}$$

The control law is obtained by solving the following modified algebraic Riccati equation

$$- S F^1 - F^{1^T} S - A + (S G^1 + N) B^{-1} (SG^1 + N)^T = 0 \tag{27}$$

and

$$u = B^{-1}(S G^1 + N)^T X \tag{28}$$

Equation (28) is written in an equivalent form as

$$u = C_1 x + C_2 z_1 + C_3 z_2 \tag{29}$$

The generalized controller structure is then shown in Figure 1. This controller has the form of a dynamic compensator. The transfer function between the control input u and the state x is

$$z_1\ (j\omega) = (j\omega I - F_1)^{-1} G_1 x \tag{30}$$

$$z_2\ (j\omega) = (j\omega I - F_2)^{-1} G_2 u \tag{31}$$

$$[I - C_3(j\omega I - F_2)^{-1} G_2] u(j\omega) = [C_1 + C_2(j\omega I - F_1)^{-1} G_1] x(j\omega)$$

or (32)

$$u(j\omega) = [I - C_3(j\omega I - F_2)^{-1} G_2]^{-1} [C_1 + C_2(j\omega I - F_1)^{-1} G_1] x(j\omega)$$

The compensator may be expressed in an equivalent form shown in Figure 2. Note that the transfer functions between x^1 and u and u^1 and u are

$$x^1(j\omega) = \left[H_1\ (j\omega I - F_1)^{-1} G_1 + D_1\right] u\ (j\omega) \tag{33}$$

$$u^1(j\omega) = \left[H_2\ (j\omega I - F_2)^{-1} G_2 + D_2\right] u\ (j\omega) \tag{34}$$

Therefore, the poles of $P_1(j\omega)$ and $P_2(j\omega)$ show up as compensator poles and zeros, respectively and directly influence the closed loop transfer function. This provides a direct relationship between the form of the frequency dependence shaping used in the cost function and the structure of the overall compensator.

If the number of zeros in $P_1(j\omega)$ exceeds the number of poles, Eq. (21) must be modified. For example, if $P_1(j\omega)$ has one more zero than poles, Eq. (21) may be written as

$$\begin{aligned} \dot{z}_1 &= F_1 z_1 + G_1 x \\ x^1 &= H_1 z_1 + D_1 x + D_{11}\dot{x} \end{aligned} \tag{35}$$

Using Eq. (1), this may be written as

$$\dot{z}_1 = F_1 z_1 + G_1 x$$

$$\begin{aligned} x^1 &= H_1 z_1 + D_1 x + D_{11} (Fx+Gu) \\ &= H_1 z_1 + (D_1 + D_{11} F)\, x + D_{11} Gu \end{aligned} \tag{36}$$

Therefore when $P_1(j\omega)$ may have k_1 more zeroes than poles, a more general form for Eq. (21) is

$$\dot{z} = F_1 z_1 + G_1 x$$

$$x^1 = H_1 z_1 + D_1 x + \sum_{i=0}^{k_1 - 1} D_1{}^i u^{(i)} \tag{37}$$

Where $u^{(i)}$ is the ith derivative of u. Equations (22), (23), and (24) are also modified similarly.

V. EXAMPLES

Example 1: Control of aircraft during landing in constant lateral winds. The equations of an aircraft are

$$\dot{x} = Fx + Gu + \Gamma w \tag{38}$$

$x^T = [v, p, r, \phi]$, $u^T = [\delta a, \delta r]$ and w is constant lateral wind (v, p, r and ϕ are lateral speed, roll rate, yaw rate and roll angle; δa and δr are lateral inputs). Control laws based on standard LQG methods are of the form

$$u = Cx \tag{39}$$

and for any finite C give a constant steady state error.

Constant biases correspond to errors at zero frequency. To reduce the steady state error in v to zero, an additional term of the following forms should be added to the performance index

$$\frac{v^2}{\omega^2} \tag{40}$$

Defining

$$\xi = \frac{v}{j\omega} \rightarrow \dot{\xi} = v \tag{41}$$

the additional term in the performance index is ξ^2. This is integral control!

There is a similar problem in many systems where the rates of change of control inputs affect system dynamics leading to control saturation. An example is the control of spacecraft attitude using control moment gyros. To avoid saturation, additional terms of the following form may be added to the performance index

$$\frac{u^2}{\omega^2} \tag{42}$$

This is input integral control to avoid saturation.

Example 2: <u>Design of a control system for an overhead crane</u> (Figure 3) The equations of motion can be approximated for small θ

$$m\ddot{x}_1 = mg\sin\theta \approx mg\frac{(x_2 - x_1)}{\ell} \tag{43}$$

$$m_1\ddot{x}_2 = f - mg\sin\theta \approx f - \frac{m_1 g}{\ell}(x_2 - x_1) \tag{44}$$

Consider a case in which $m = m_1 = 1{,}000$ lbm, $\ell = 32$ ft and $g = 32$ ft sec^{-2}. The open loop eigenvalues are at 0.0, 0.0, $\pm\sqrt{2}j$. The problem is to design a regulator to control the position of mass hanging from the crane without producing serious residual oscillations.

A standard cost functional is of the form

$$J = \lim_{T\to\infty} \frac{1}{T} \int_0^T (a_1 x_1^2 + a_2 x_2^2 + u^2)dt \tag{45}$$

A good response is obtained for $a_1 = 1$ and $a_2 = 0$. The time histories of x_1, x_2 and u are shown in Figure 4. Notice that the hanging mass continues to oscillate at about $\sqrt{2}$ rad sec^{-1}. Improved response can be obtained by placing some penalty on velocities.

The most improvement was obtained by using frequency shaping methods. Since there is significant oscillation around $\sqrt{2}$ rad sec^{-1}, it is useful to include a term of the following form in the performance index.

$$\frac{\ell^2\theta^2}{(\omega^2-2)^2} \tag{46}$$

This term is included with a unity weighting. The resulting controller transfer function is

$$u(s) = \left[-.64s + 1.1 + \frac{.85 + .38s}{s^2 + 2}\right]x_1(s) + \left[-2.2s - 2.1 + \frac{.85 + .38s}{s^2 + 2}\right]x_2(s) \tag{47}$$

Time history of responses starting from an initial condition error of 1m are shown in Figure 5. Notice that the maximum

control input is the same. Major differences in the applied control input occur throughout the trajectory. At the end of 10 sec., the amplitude of oscillation is an order of magnitude smaller than it is in Figure 4. This shows the effectiveness of frequency shaping methods.

Example 3: <u>Vibration control of a helicopter.</u> Helicopters suffer from significant vibration at $N\Omega$, $2N\Omega$ ... etc., discrete frequencies where N is the number of blades in the rotor and Ω is the ratational frequency. In the example considered here, the equations of motion describing the behavior of a helicopter near the vibration frequency are assumed to be

$$\begin{bmatrix} \dot{w} \\ \dot{\beta}_0 \\ \ddot{\beta}_0 \end{bmatrix} = \begin{bmatrix} -0.0839 & -241.4 & -11.35 \\ 0 & 0 & 1 \\ 0.1186 & -142.7 & -1.157 \end{bmatrix} \begin{bmatrix} w \\ \beta_0 \\ \dot{\beta}_0 \end{bmatrix} + \begin{bmatrix} -2.82 \\ 0 \\ 2.2 \end{bmatrix} \delta_{coll}$$

$$+ \begin{bmatrix} 0.1 \\ 0 \\ -0.2 \end{bmatrix} \delta_{VIB} \qquad (48)$$

where

w vertical speed (ft/sec)

β_0 coning angle of the blades (rad)

δ_{coll} is the collective pitch control (inches)

δ_{VIB} is the oscillating disturbance (force)

consider vibration at 20 rad $\sec^{-1}$

$$\delta_{VIB} = 0.2 \cos (20\, t) \tag{49}$$

where t is time in seconds.

The way to avoid vibration is to minimize response at $\omega_V = 20$ rad $\sec^{-1}$. A frequency shaped cost functional of the following form may be selected.

$$J = \frac{1}{2\pi} \int_{-\infty}^{\infty} Q_2 |P(j\omega)|^2 \, |w\,(j\omega)|^2 + R\,|\delta_{coll}(j\omega)|^2 \, d\omega \tag{50}$$

where

$$P(j\omega) = \frac{j400\omega}{400-\omega^2+j40\xi\omega} \tag{51}$$

And ξ is a design parameter. The cost functional is converted into a time domain formulation by adding the following states.

$$\frac{d}{dt} \begin{bmatrix} y_I \\ \dot{y}_I \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -400 & \xi \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 400 \end{bmatrix} w \tag{52}$$

This equation, combined with the helicopter model, specifies the plant description to be used with the performance index described in equation (4). The augmented state description is:

$$\begin{bmatrix} \dot{W} \\ \dot{\theta} \\ \ddot{\theta} \\ \hline \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \left[\begin{array}{ccccc} -0.0839 & -241.4 & -11.23 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0.1186 & -142.7 & -1.137 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \\ 400 & 0 & 0 & -400. & -40 \end{array}\right] \begin{bmatrix} W \\ \theta \\ \dot{\theta} \\ \hline z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} [illegible] \\ 0 \\ [illegible] \\ \hline 0 \\ 0 \end{bmatrix} \delta_{coll} \tag{53}$$

The optimal control law is of the form:

$$\delta_{coll} = C_W W + C_\theta \theta + C_{\dot{\theta}} \dot{\theta} + C z_1 + C_2 z_2 \tag{54}$$

An implementation of this vibration controller is shown in Figure 6. The vibration level at a point in the helicopter after the controller is switched on is shown in Figure 7 (ξ=.01). Even though ξ can be set to zero, a small value of ξ was used to minimize sensitivity to parameter variations.

Example 4: Control of Structures. Dynamics of structures is described by a set of partial differential equations. Finite order model representations are valid only at low frequencies. We show how frequency chaping methods are used to design control laws for models valid only at low frequencies.

Consider a free standing pyramid where we wish to control line-of-sight errors at the apex (Fig. 8). The size of various elements are given in Tables 1 and 2.* We have an eight mode model representing the behavior of the system below 1 H_z. Three actuators and three sensors are available to actively increase damping ratios of specific modes which affect open line-of-sight errors. All modes have an open loop damping ratios of 0.5%.

---

* This example was developed by Dr. Robert Strunce and his associates at Charles Stark Draper Laboratory, Cambridge, Mass.

Since the model is valid only at low frequencies, it is desirable to minimize control and state activities above 1 Hz. This may be achieved by a control weighting matrix which increases with frequency, for example:

$$B(j\omega) = A_B(\omega^2+\omega_o^2) \tag{55}$$

A_B is a diagonal matrix with elements b_1, b_2, and b_3. By selecting ω_o at 0.5 Hz, the control weighting at 1.5 Hz is five times higher than at 0.5 Hz. The control weighting is written equivalently as:

$$(u^T A_B u)(\omega^2+\omega_o^2) = v^T A_B v \tag{56}$$

where

$$\dot{u} + \omega_o u = v \tag{57}$$

The line-of-sight control requires increased damping in modes 1,2, 4 and 5. This is achieved by state weighting of the form

$$0.5\ (q_1^2 + q_2^2 + q_4^2 + \dot{q}_1^2 + \dot{q}_2^2 + \dot{q}_4^2)$$
$$+\ (q_5^2 + \dot{q}_5^2) \tag{58}$$

q_i is the modal deflection corresponding to the i th mode. b_1, b_2 and b_3 are all set at .01.

Table 3 shows the closed loop eigenvalues for the 8 mode design model and the 12 mode evaluation model. Because of the low control activity at high frequency, the damping ratios of unmodeled, high frequency modes have small perturbations.

When a comparable design was carried out without frequency shaping of state and control weighting matrices, the perturbations in the unmodeled eigenvalues was so large that some of the modes were unstable.

Discussion: We have shown some simple applications of frequency shaping methods. The method is much more powerful and is being applied to many complex problems.

A reviewer provides an interesting interpretation of the controllers of Examples 2, 3 and 4. Example 2 and 3 give "virtual tuned vibration absorber" while Example 4 gives a "virtual actuator dynamics".

VI. CONCLUSIONS

Frequency shaping of quadratic cost functionals can provide additional flexibility in developing control laws using modern control theory techniques. For good design, the particular selection of weighting functions depends on (a) model validity, (b) actuator/sensor characteristics, (c) control objectives, and (d) disturbance spectrum. Examples were presented to demonstrate the effectiveness of the proposed approach.

The design procedure for frequency shaped weighting on state and control law is implemented by linear elements with memory.

The frequency shaping concept has been extended for state estimation with models valid within a finite frequency range and with finite bandwidth sensors. These ideas may be also be used in parameter identification.

VII. ACKNOWLEDGEMENTS

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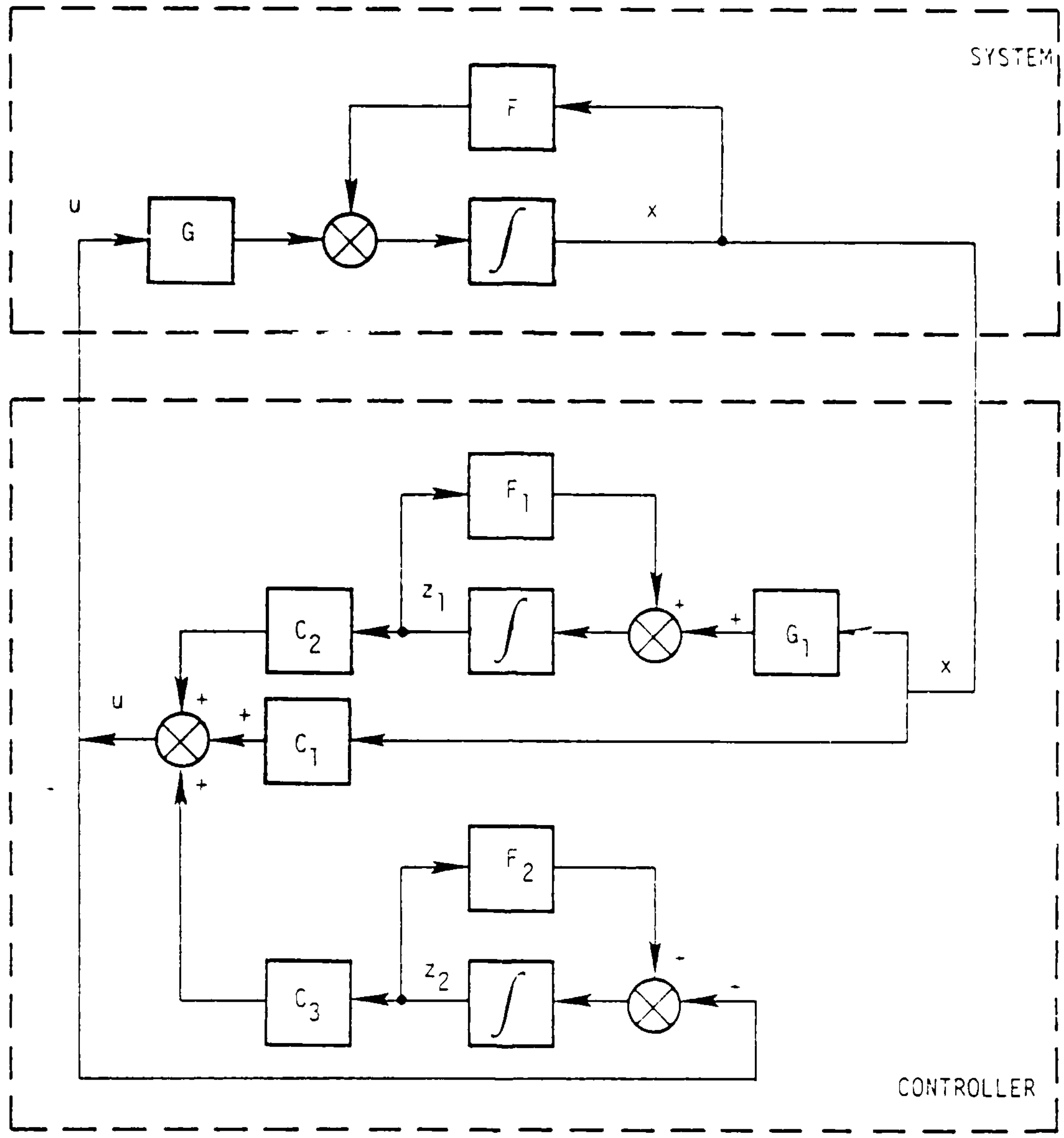


Figure 1 Structure of the Generalized Controller

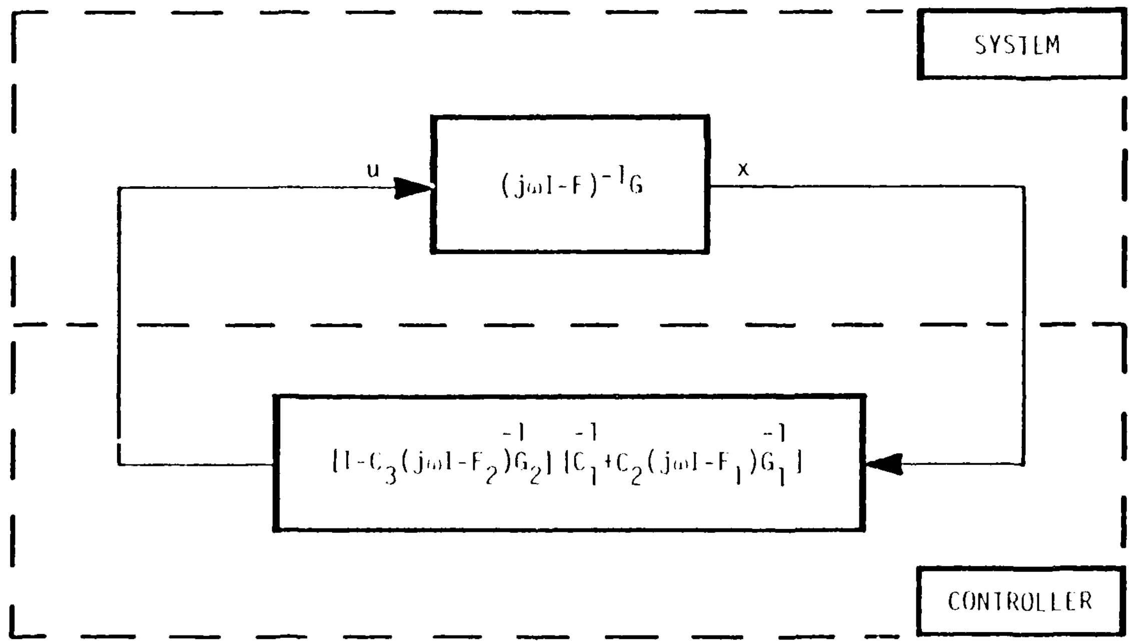


Figure 2 An Alternate Representation of a Frequency Shaped Controller

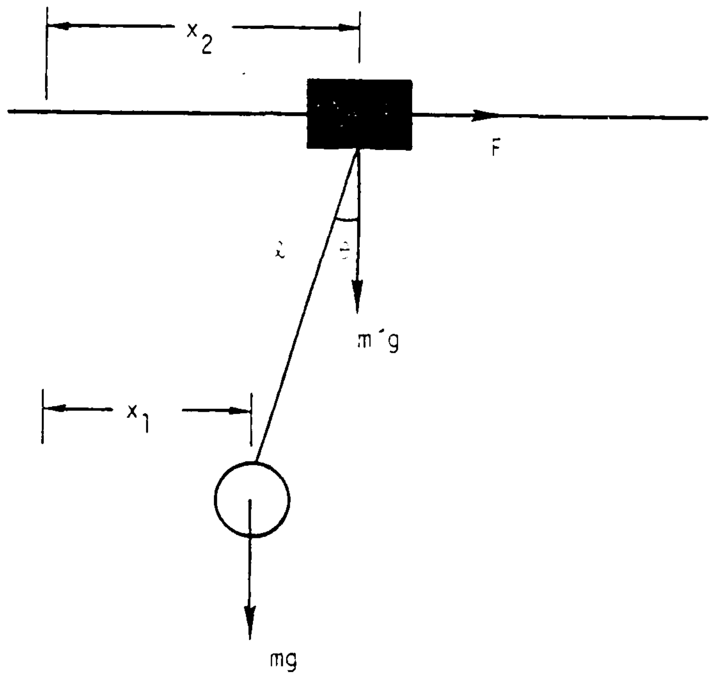


Figure 3 Dynamics of an Overhead Crane

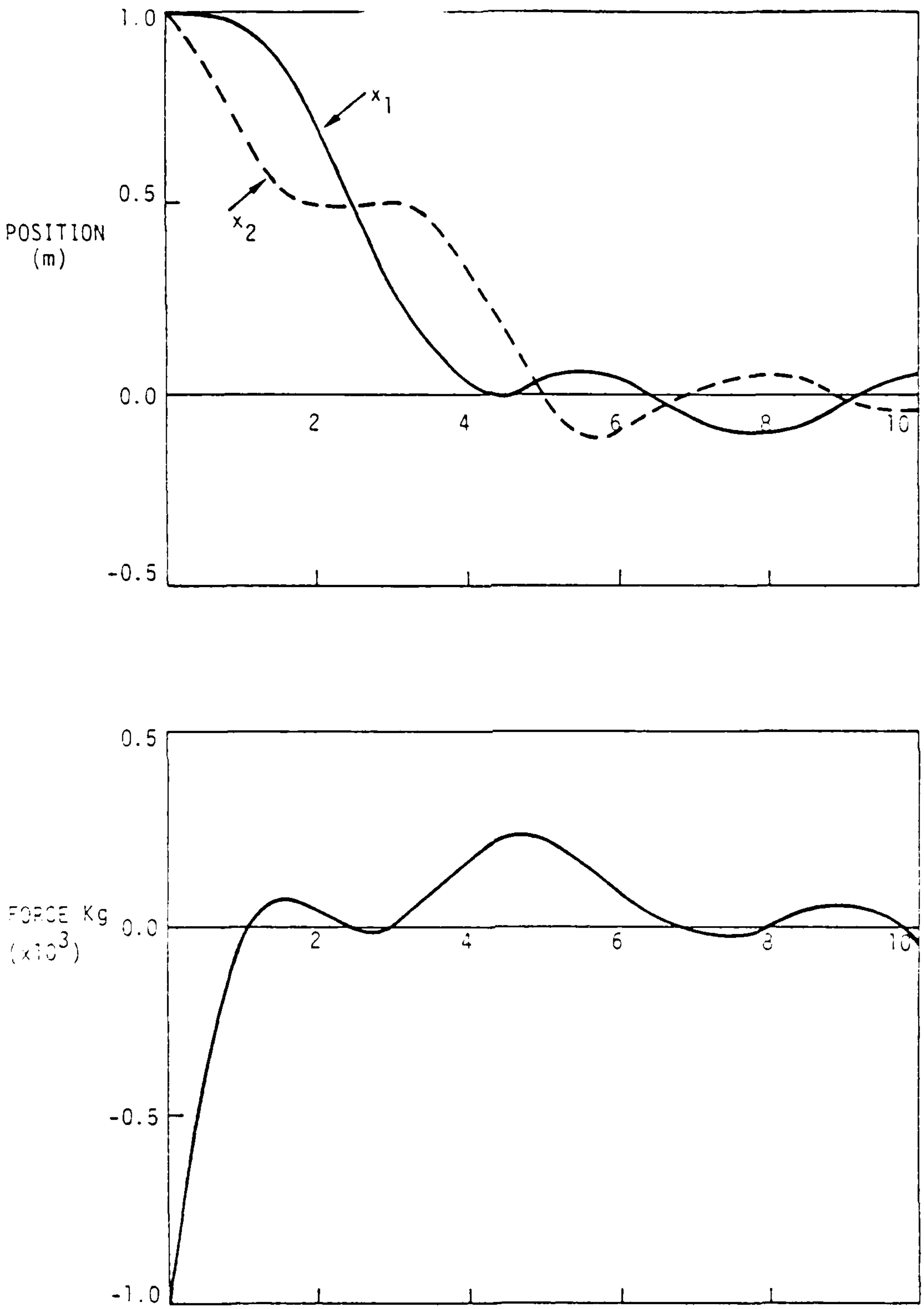
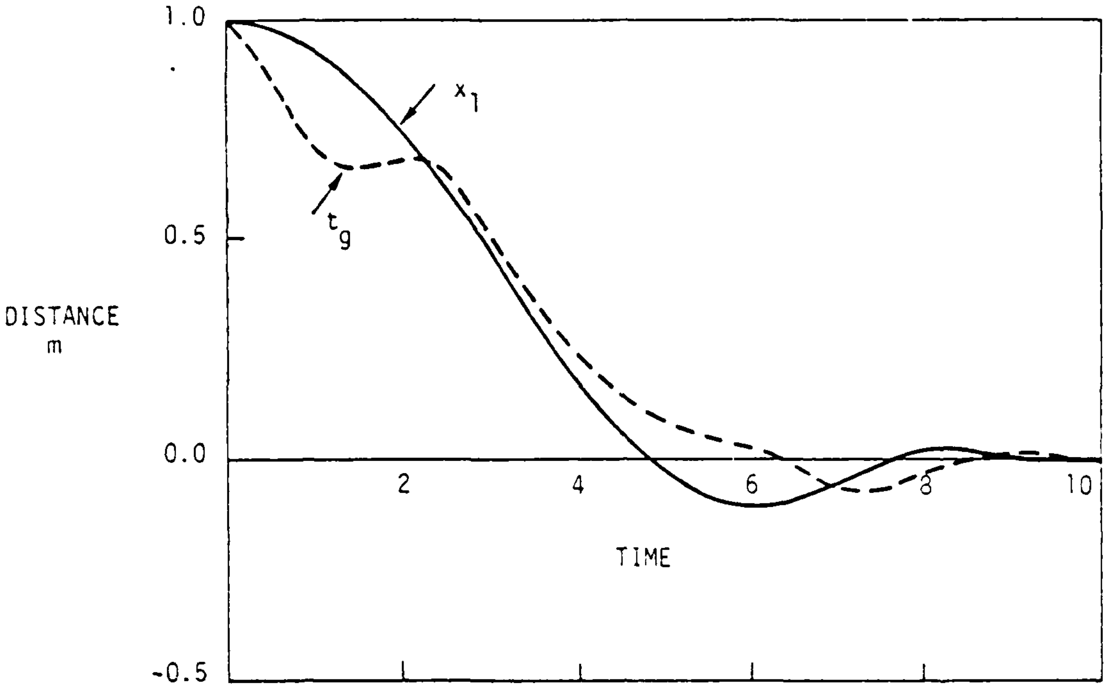


Figure 4 Time Histories of Responses with Cost Functional $(x_1^2+u^2)$



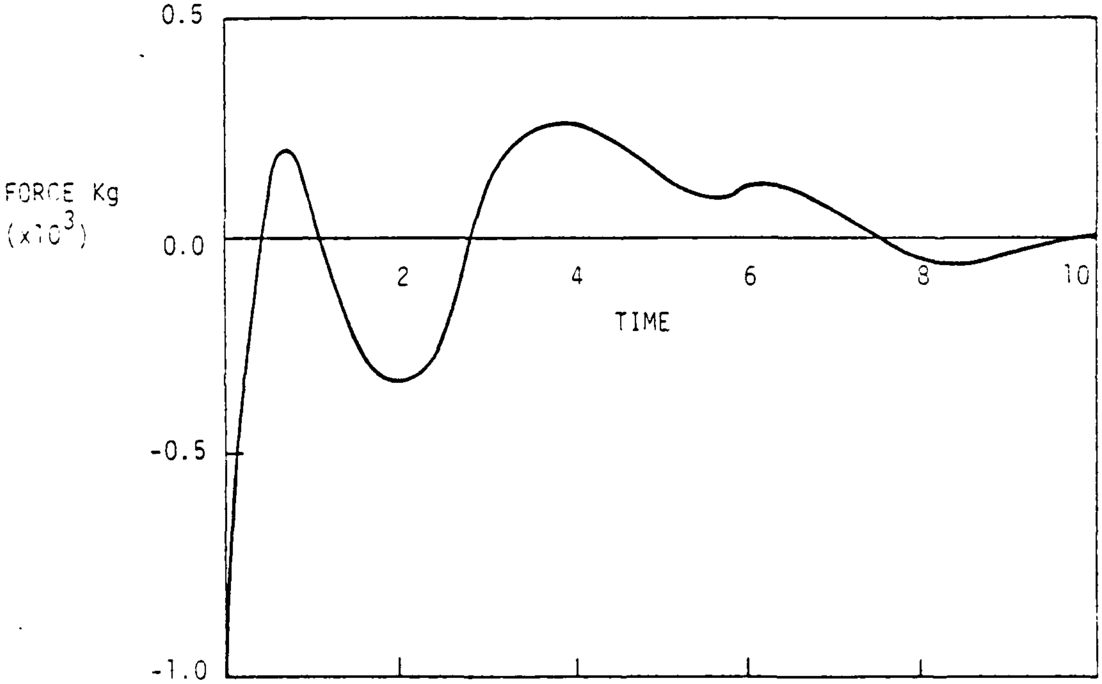


Figure 5 Overhead Crane Response Time Histories with Frequency Shaped Cost Functional

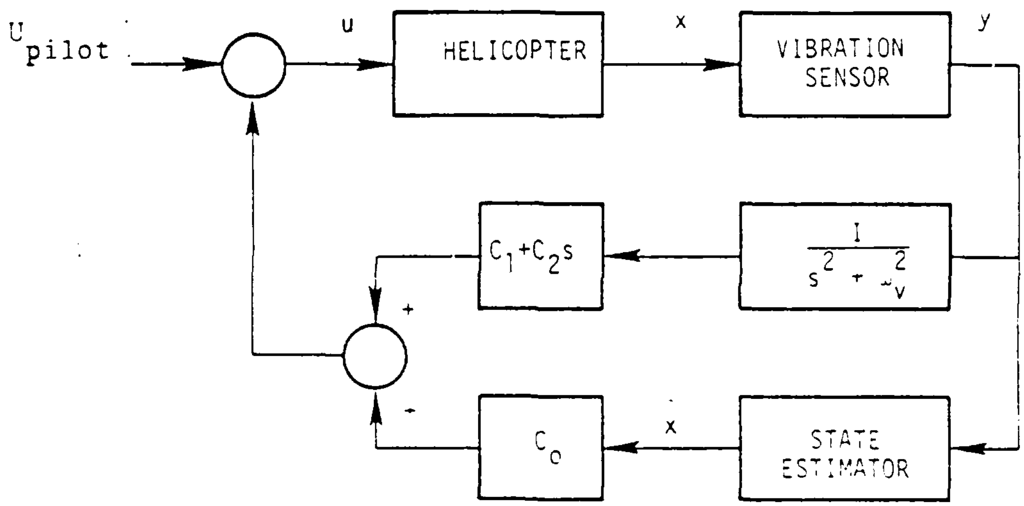


Figure 6 Structure of Helicopter Vibration Controller

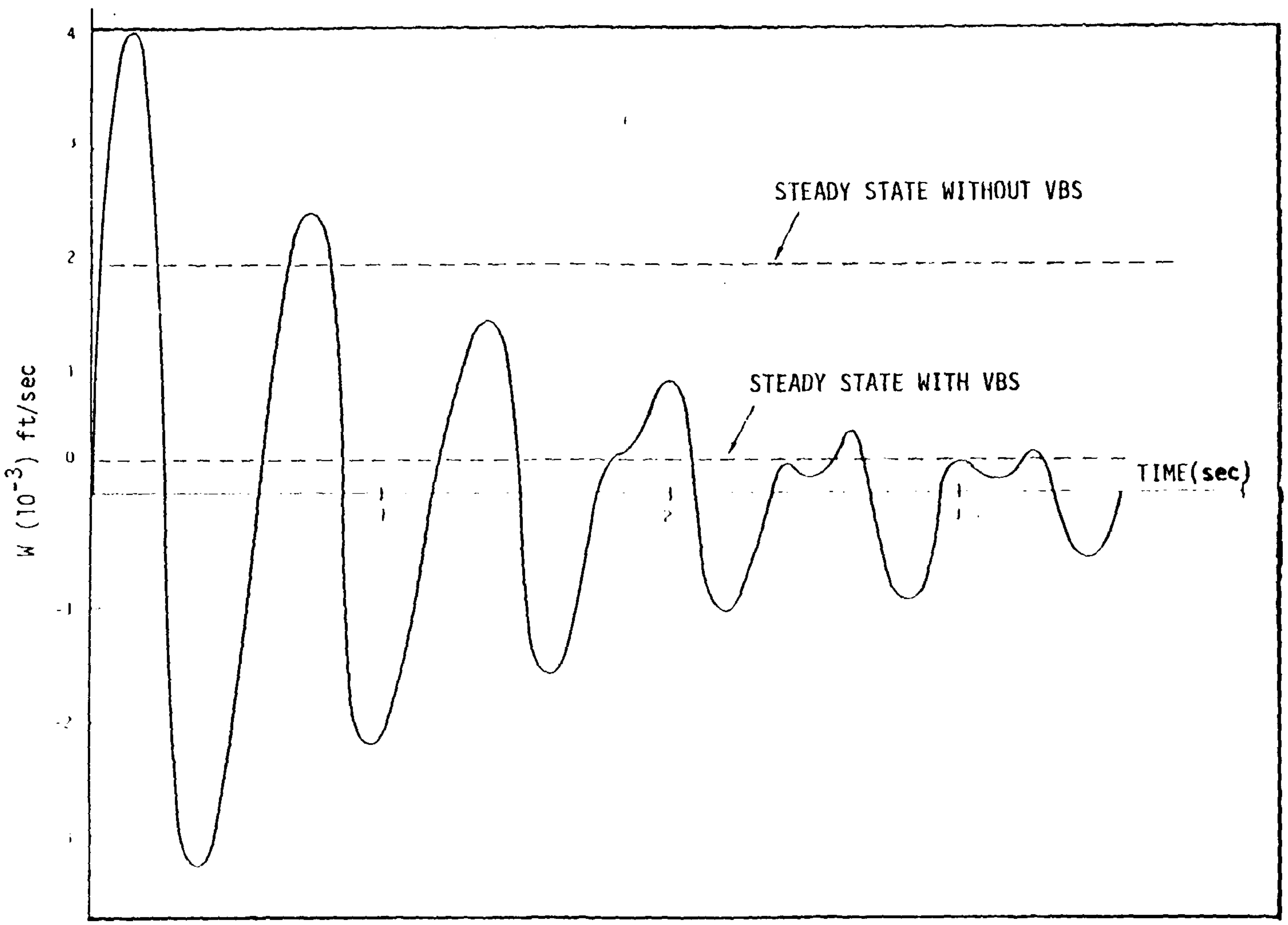


Figure 7 Vertical Vibration vs Time with Vibration Suppresion Regulator

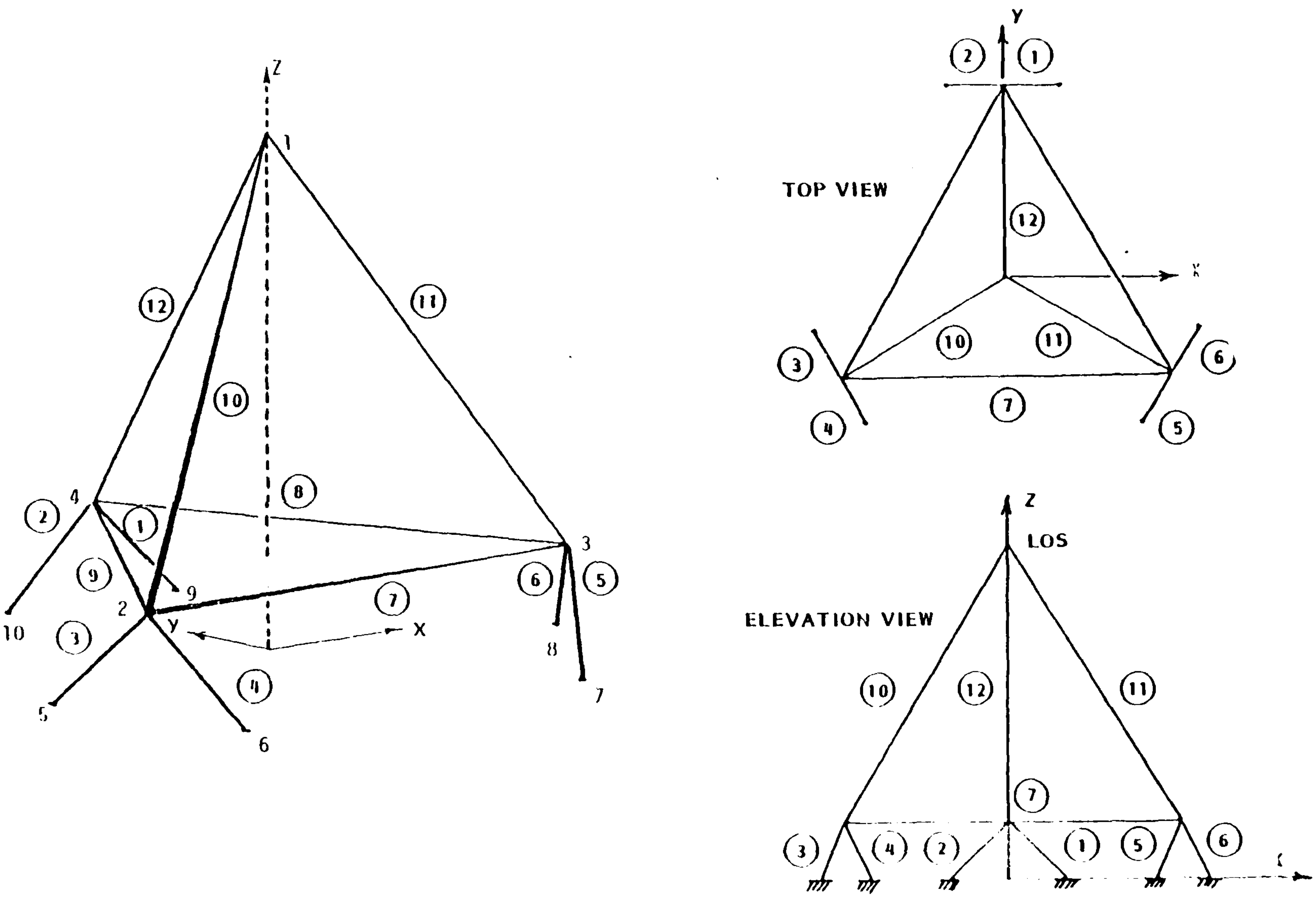


Figure 8 The CSDL Example Structural Model

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Table 1
Cross Sectional Areas

TRUSS ELEMENT NO.	AREA
1	100
2	100
3	100
4	100
5	100
6	100
7	1000
8	1000
9	1000
10	1000
11	100
12	100

Table 2
Structural Node Coordinates

NODE	X	Y	Z
1	0.0	0.0	10.165
2	-5.0	-2.887	2.0
3	5.0	-2.887	2.0
4	0.0	5.7735	2.0
5	-6.0	-1.1547	0.0
6	-4.0	-4.6188	0.0
7	4.0	-4.6188	0.0
8	6.0	-1.1547	0.0
9	2.0	5.7735	0.0
10	-2.0	5.7735	0.0

APPENDIX C: THE ACCOMODATION OF LARGE PLANT PARAMETER VARIATIONS BY STATE FEEDBACK

This section examines the stability properties of a closed-loop system under a state feedback control law and in the presence of parameter variations in the system's matrix. It is shown that if the variations are of a special class, the stability properties of the closed-loop system will be insensitive to these variations. This class of systems can be made even larger if a certain given matrix is stable. This analysis distinguishes systems where linear state feedback control laws work adequately and systems where more detailed adaptive control laws are required.

I. INTRODUCTION

In many control problems, knowledge of plant parameters is necessary in order for a particular design procedure to be carried out. In practice, however, the parameters of the system may change under operating conditions which will cause them to acquire uncertain values. These uncertain, or inaccurate values of parameters may be present from the beginning due to modeling error. The design of a stabilizing control law, or any solution of the many possible optimal control problems, should take these variations, or uncertainties in parameters, into consideration. A few different approaches to this problem can be found in [1] through [5].

This paper examines the stability properties of a linear dynamic system whose parameters may change within a known range, where this range is not restricted to be necessarily small. It will be shown that one can identify a class of systems that can exhibit high insensitivity to parameter variations.

The structure of the paper is as follows. Section II contains the statement of the problem to be considered, and Section III presents the modalator approach toward the pole placement problem by state feedback. The structure offered by this approach is used in Section IV where the sensitivity of the stability properties to parameter variations is examined. Section V presents our conclusions.

II. PROBLEM STATEMENT

Consider the following linear, continuous-time dynamic system described by

$$\dot{x}(t) = A_o x(t) + Bu(t) \qquad (2.1)$$

where $x \in R^n$, $u \in R^m$ are the state and input vectors respectively, A_o is an $n \times n$ matrix and B is an $n \cdot m$ $(m < n)$ matrix of full rank. Furthermore, it is assumed that the input matrix B remains constant but the parameters of the matrix A_o are subject to changes over a (possibly large) known range. These variations of parameters are either induced under operating conditions or are simply due to error introduced during the modeling process. In either case, the parameters of the matrix A_o are bounded within the known range of possible parameter variations, namely

$$A(\min) \leq A_o \leq A(\max) \qquad (2.2)$$

where the inequalities are taken component-wise. Another possible way of representing the matrix A_o is by

$$A_o = A + A_1 \qquad (2.3)$$

where A is the design (or nominal) matrix and A_1 is the mat x of deviations from the design values. It should be pointed out, however, that while A is taken as a known, constant matrix, A_1 can change over the complete range of possible parameter variations and its value will not normally be known to the designer.

The problem to be considered in this paper is as follows. Given the dynamic system described in Eq. (2.1) with its A_o

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matrix as in Eqs. (2.2) and (2.3). Can one find a constant-gain state feedback control law that will keep the closed-loop system stable over the complete range of possible parameter variations? For the problem to be solved successfully, it should specify the conditions under which it is solvable and, also, identify the class of A_1 matrices for which the stability properties of the closed-loop system will be insensitive to parameter variations. Ideally, one would like to design a closed-loop system that will be insensitive even to large parameter variations.

The next section will present a particular method for the derivation of the state feedback control law. The structure offered by this method will prove very useful in answering the questions posed above.

III. STATE FEEDBACK

A new way toward the output feedback problem was developed in [9], introduced in [10], and later extended to state feedback problems in [11]. This new approach was termed the <u>modalator</u> approach since the solution of the pole-placement problem is obtained via a particular dynamic system that provides information on the <u>modal</u> structure of the closed-loop system under the particular state (or output) feedback control law.

For convenience and easy reference, the main modalator result for pole-placement by state feedback will be given below, [11].

<u>Note</u>: Prime will denote transposition, i.e., A' is the transpose of A .

THEOREM 3.1 (MAIN MODALATOR RESULT)

Consider the linear, controllable, continuous-time* dynamic system described by

$$\dot{x}(t) = Ax(t) + Bu(t) \ , \ x \in \mathbb{R}^n, \ u \in \mathbb{R}^m \ , \ m<n \ . \qquad (3.1)$$

Then, one can always find a stable $(n-m)\times(n-m)$ matrix F with arbitrary eigenvalues and an $(n-m)\times m$ matrix H such that the $(n-m)xn$ transformation matrix Q that solves

$$QA' - F'Q = HB' \qquad (3.2)$$

*This procedure is applicable to discrete-time dynamic systems as well.

allows

$$\begin{bmatrix} B' \\ \hdashline Q \end{bmatrix}^{-1} \triangleq \begin{bmatrix} \underbrace{D}_{m} & \vdots & \underbrace{E}_{n-m} \end{bmatrix} . \tag{3.3}$$

Then, the following will hold:

(i) $$F' = QA'E \tag{3.4}$$

$$H = QA'D . \tag{3.5}$$

(ii) If the state feedback control law is given by $u(t) = Kx(t)$ where

$$K = -D'(A+\alpha I_n) \ , \ \alpha>0 \tag{3.6}$$

then the closed-loop matrix, $\hat{A}$, given by

$$\hat{A} = Q'E'A - \alpha BD' \tag{3.7}$$

will have $n-m$ of its eigenvalues equal to those of the matrix F and the remaining m closed-loop eigenvalues will be clustered at $(-\alpha)$, i.e.,

$$\pi(\hat{A}) \triangleq |\hat{A}-\lambda I_n| = (\lambda+\alpha)^m \ |F-\lambda I_{n-m}| = 0. \tag{3.8}$$

<u>Proof</u>: The statements regarding the solution of Eq. (3.2), the inversion of Eq. (3.3), and the expressions given in Eqs. (3.4 and (3.5) are well known, (see, e.g., [12] and [13]).

If one is using the state feedback gain matrix given by Eq. (3.6), then the closed-loop matrix, $\hat{A}$, is given by

$$\hat{A} \triangleq A+BK = (I-BD')A-\alpha BD'$$

but, using Eq. (3.3), this becomes

$$\hat{A} = Q'E'A-\alpha BD' .$$

The closed-loop eigenvalues are found from

$$\pi(\hat{A}) = |Q'E'A-\alpha BD'-\lambda I_n| = |Q'E'(A+\alpha I_n)-(\lambda+\alpha)I_n| = 0$$

This can be simplified [14] to yield

$$\pi(\hat{A}) = (\lambda+\alpha)^m |E'AQ'-\lambda I_{n-m}| = (\lambda+\alpha)^m |F-\lambda I_{n-m}| = 0 .$$

∎

This procedure for the derivation of the state feedback control law results in a special property that can be observed by examining the proof of theorem 3.1 and will be stated here explicitly.

<u>Corollary 3.1 (Separation Property)</u>: The choice of the scalar α in the derivation of the state feedback gain matrix in Eq. (3.6) will not affect the choice made for the n-m eigenvalues of the matrix F in Eq. (3.2). ∎

The importance of this corollary lies with the fact that once F is determined, changing the scalar α will have no effect on the eigenvalues of F but will only move the m clustered eigenvalues along the real axis. This property is realizable because the modalator approach, summarized in theorem 3.1, factors the n-dimensional state space into two invariant subspaces.

The procedure presented here involved a certain matrix equation whose solution provides the state feedback gain matrix in a closed-form expression. This matrix equation is usually associated with the Luenberger observer [15], and it is of interest to examine the relation, if any, between the observer and the modalator approach. This relation will be examined through the concept of duality.

The system in Eq. (3.1) given by

$$\begin{cases}\dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = x(t)\end{cases}$$

has a dual system given by

$$\begin{aligned}\dot{x}^*(t) &= A'x^*(t) + u^*(t) \\ y^*(t) &= B'x^*(t) \ .\end{aligned} \tag{3.9}$$

It can be easily verified that if

$$u^*(t) = -(A'+\alpha I_n)DB'x^*(t) \tag{3.10}$$

then the vector defined by

$$z(t) \triangleq Qx^*(t) \tag{3.11}$$

evolves dynamically according to

$$\dot{z}(t) = F'z(t) \ . \tag{3.12}$$

Furthermore, it can be shown that the transformation matrix, Q , in addition to solving $QA'-F'Q = HB'$, also solves

$$Q\hat{A}' - F'Q = 0 \ . \tag{3.13}$$

Equation (3.13) is particularly interesting because it reveals that the matrix Q is a matrix expressible by eigenvectors of $\hat{A}$ that correspond to the common eigenvalues of $\hat{A}$ and F . The modal structure revealed through Q motivates the terminology chosen for the modalator. Furthermore, this structure explains the absence of an output term in Eq. (3.12). We will not elaborate this point any further; the interested reader is referred to [10] and [11].

IV. STABILITY PROPERTIES AND PARAMETER VARIATIONS

The objective of this paper, as stated in Section II, is the derivation of a state feedback control law that ensures the stability of the closed-loop system even in the presence of parameter variations. Therefore, one would like to design an appropriate state feedback control law such that the actual closed-loop matrix given by

$$\hat{A}_o = \hat{A} + A_1 \tag{4.1}$$

where

$$\hat{A} \triangleq A + BK \tag{4.2}$$

will remain stable over all possible variations in A_1. Obviously, this problem may not be solvable for every case and, therefore, one should indicate the class of systems for which the problem is solvable.

The $n \times m$ input matrix B was assumed to be of full rank equal to m. Therefore, without loss of generality, we can consider the matrix B as given in the form

$$B = \begin{bmatrix} I_m \\ \hline 0 \end{bmatrix}, \tag{4.3}$$

where I_m is the $m \times m$ identity matrix. When B is not in the form of Eq. (4.3), one can employ a simple transformation that yields an equivalent system whose input matrix is in the form of Eq. (4.3). The details of this transformation can be found in Appendix A.

Since the matrix $\begin{bmatrix} B' \\ \hline Q' \end{bmatrix}$ of Eq. (3.3) is nonsingular, the columns of B and Q' can serve as a basis for the n-dimensional

state space, hence the matrix A_1 can be written as

$$A_1 = BR + Q'T \tag{4.4}$$

where

R is an $m \times n$ matrix

T is an $(n-m) \times n$ matrix.

The matrices R and T are determined by the (given) range of possible parameter variations. The actual values of these matrices, however, are not known to the designer.

Using Eqs. (4.1), (4.4) and (3.7), the actual closed-loop matrix is given by

$$\hat{A}_o = Q'E'A - \alpha BD' + BR + Q'T . \tag{4.5}$$

Since $BD' + Q'E' = I_n$, Eq. (4.5) becomes

$$\hat{A}_o = Q'E'(A+\alpha I_n) - \alpha I_n + BR + Q'T$$

which can be written compactly as

$$\hat{A}_o = \left[B \,\vdots\, Q' \right] \left[\frac{R}{E'(A+\alpha I_n)+T} \right] - \alpha I_n . \tag{4.6}$$

A question of interest will be the following one. Can one specify a class of R and T matrices that allows the stabilization of A_o over all parameter variations? A sufficient condition for an affirmative answer is given below.

<u>Theorem 4.1</u>: If the procedure of theorem 3.1 is used in the derivation of the state feedback gain matrix, and if

$$T' \varepsilon N(Q) \tag{4.7}$$

$$R' \varepsilon N(Q) \tag{4.8}$$

where $N(Q)$ denotes the null space of Q,

then the actual closed-loop matrix, $\hat{A}_o$, will have n-m of its eigenvalues equal to those of F and m eigenvalues equal to those of the matrix $(RB-\alpha I_m)$, i.e.,

$$\pi(\hat{A}_o) = |(RB-\alpha I_m) - \lambda I_m| \cdot |F-\lambda I_{n-m}| = 0 .$$

<u>Proof</u>: From Eq. (4.6) we find

$$\pi(\hat{A}_o) = \left| \begin{bmatrix} R \\ \hline E'(A+\alpha I_n)+T \end{bmatrix} \begin{bmatrix} B & \vdots & Q' \end{bmatrix} - (\lambda+\alpha) I_n \right| = 0.$$

Using Eqs. (3.3) and (3.4), this becomes

$$\pi(\hat{A}_o) = \left| \left[\begin{array}{c|c} RB & RQ' \\ \hline E'(A+\alpha I_n)B+TB & F+\alpha I_{n-m}+TQ' \end{array}\right] - (\lambda+\alpha) I_n \right| = 0 .$$

However, if R and T are such that Eqs. (4.7) and (4.8) hold, then

$$\pi(\hat{A}_o) = |(RB-\alpha)-\lambda I_m| \cdot |F-\lambda I_{n-m}| = 0 .$$ ■

Note that if Eqs. (4.7) and (4.8) hold, then the eigenvalues of F will not change. Equation (4.9) also indicates what is the role of the positive scalar α .

<u>Corollary 4.1</u>: If the positive scalar, α , is chosen such that

$$\alpha > \lambda_{max}(RB) \tag{4.10}$$

then the actual closed-loop system will remain stable for all parameter variations satisfying Eqs. (4.7) and (4.8). ■

The results presented above show that n-m actual closed-loop eigenvalues will be insensitive to changes in A_1 and that the remaining m closed-loop eigenvalues can be ensured to remain stable by a proper choice of α, provided the R and T matrices are in the null space of the transformation matrix Q . It may be argued, however, that since one starts the design with a given range of possible parameter variations, it may not be possible to find an appropriate transformation matrix Q that will satisfy Eqs. (4.7) and (4.8). This point is, of course, valid; and we will therefore turn our attention to identify a class of systems whose given variation will satisfy Eqs. (4.7) and (4.8) and, therefore, will be accommodated by a constant-gain state feedback control law.

It is well known (see, e.g., [13]) that if one starts with an invertible matrix in Eq. (3.3), then the solution of Eq. (3.2) amounts to a pole-placement problem for the matrix F. Therefore, if one partitions A according to

$$A = \begin{array}{cc} \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right] & \begin{array}{l} \}\ m \\ \}\ n-m \end{array} \\ \begin{array}{cc} \underbrace{\quad}_{m} & \underbrace{\quad}_{n-m} \end{array} & \end{array} \tag{4.11}$$

and if

$$Q = \left[Q_1 \,\vdots\, I_{n-m} \right] , \tag{4.12}$$

then, if $B' = \left[I_m \,\vdots\, 0 \right]$, it can be shown that

$$F = E'AQ' = A_{22} + A_{21}\, Q_1' \tag{4.13}$$

and the purpose of Q_1 is to achieve the desired pole-placement for F. Then

$$\begin{bmatrix} B' \\ \hdashline Q \end{bmatrix}^{-1} = \begin{bmatrix} I_m & \vdots & 0 \\ \hdashline Q_1 & \vdots & I_{n-m} \end{bmatrix}^{-1} = \begin{bmatrix} I_m & \vdots & 0 \\ \hdashline -Q_1 & \vdots & I_{n-m} \end{bmatrix} \doteq \begin{bmatrix} D & \vdots & E \end{bmatrix} . \tag{4.14}$$

We are now in a position to identify the class of A_1 matrices that satisfy Eqs. (4.7) and (4.8).

Let

$$R = [\underbrace{R_1}_{m} \vdots \underbrace{R_2}_{n-m}] \tag{4.15}$$

$$T = [\underbrace{T_1}_{m} \vdots \underbrace{T_2}_{n-m}] \tag{4.16}$$

then, Eqs. (4.7) and (4.8) requires that

$$R_2 = -R_1 Q_1'$$

$$T_2 = -T_1 Q_1' . \tag{4.17}$$

The matrix A_1 can now be written as

$$A_1 = BR + Q'T = \begin{bmatrix} R_1 + Q_1' T_1 & \vdots & -(R_1 + Q_1' T_1) Q_1' \\ \hline T_1 & \vdots & -T_1 Q_1' \end{bmatrix} = \begin{bmatrix} R_1 + Q_1' T_1 \\ \hline T_1 \end{bmatrix} \begin{bmatrix} I_m & \vdots & -Q_1' \end{bmatrix}$$

and using Eq. (4.14), this becomes

$$A_1 = \left[\frac{R_1 + Q_1' T_1}{T_1}\right] D' . \tag{4.18}$$

This shows that as long as A_1 has the form given in Eq. (4.18), the actual closed-loop system will have $n-m$ eigenvalues that are insensitive to the changes in A_1 and m eigenvalues are those of the matrix

$$F_1 \triangleq RB - \alpha I_m$$

which in this case reduces to

$$F_1 = R_1 - \alpha I_m . \tag{4.19}$$

The class of systems satisfying Eq. (4.18) is obviously quite restricted. However, close examination of Eq. (4.18) indicates a wider class of system whose stability properties will be insensitive to parameter variations.

<u>Theorem 4.2</u>: If the $(n-m)\times(n-m)$ matrix A_{22} in Eq. (4.11) is stable, and one selects the matrix F such that

$$F = A_{22} \tag{4.20}$$

then the actual closed-loop system will be completely insensitive to variations in A_{21}, and any variations in A_{11} can be accommodated by choosing α such that

$$F_1 = R_1 - \alpha I_m \tag{4.21}$$

is stable.

Proof: Notice that if A_{22} is stable and its eigenvalues are chosen for F then $Q_1 = 0$ and, from Eq. (4.18)

$$A_1 = \left[\begin{array}{c|c} R_1 & 0 \\ \hline T_1 & 0 \end{array}\right]$$

where R_1 and T_1 are arbitrary. As was shown before, $n-m$ eigenvalues will be those of F and they will be insensitive to parameter variations, while the remaining m eigenvalues are those of F_1 that can always be made stable by choosing α large enough. ■

Remark: If the input matrix, B, is not in the form given by Eq. (4.3), then the stability properties of the closed-loop matrix will not be determined by the matrix A_{22}, but a different matrix. The details can be found in the appendix.

At this juncture, a comment is in order regarding the choice of the design matrix A in Eq. (4.3). It is obvious that different choices will result in different A_1. If one chooses the design matrix, A, such that

$$A = A(\min) \tag{4.22}$$

then

$$A_1 \geq 0 \tag{4.23}$$

where the inequalities are taken component-wise. Since the matrix A_1 is non-negative, bounds can be found on its eigenvalues by either rows, or columns sums. In that case, α can be chosen to exceed these bounds and then one bypasses the need to evaluate the largest eigenvalue of the matrix R_1, as is required by Eq. (4.10). It should be pointed out, however, that

whatever choice is made for the design matrix A , it should be such that (A,B) will be a controllable pair.

The design procedure presented thus far exhibits two properties that should be pointed out explicitly. Both have to do with the dimension of the input vector. First, it is seen that the larger this dimension is, more parameters can vary without affecting the stability properties, and more variations can be taken care of by a proper choice of the single scalar α . Second, the larger this input dimension is, more closed-loop eigenvalues will be clustered on the real axis. An interesting case occurs when the state and the input vectors are of equal dimensions, i.e., $m=n$. In this case, the modalator approach, as given by theorem (3.1), seems to break down since there is no matrix equation like (3.2) to be solved. This, however, is not the case, since when $m=n$, all closed-loop eigenvalues are clustered at one point on the real axis. The state feedback gain matrix is then given simply by

$$K = -B^{-1}(A+\alpha I_n) \ , \ \alpha>0 \ . \tag{4.24}$$

This results in

$$\hat{A} = A+BK = -\alpha I_n \tag{4.25}$$

and

$$\hat{A}_o = -\alpha I_n + A_1 \ . \tag{4.26}$$

It is obvious that in this case $(m=n)$, one can always find a scalar α that will accommodate any variations introduced through A_1 .

The conditions imposed by Eqs. (4.7) and (4.8) resulted in the invariance of the eigenvalues of F to changes in A_1 . Then, if A_{22} was stable, one could design a constant-gain

state feedback control law that will yield closed-loop stability properties that are insensitive to parameter variations. The class of systems to be treated can be enlarged by dropping condition (4.7). In this case, (n-m) eigenvalues of the closed-loop system will be found from

$$\pi(F_o) \triangleq |F+TQ^* - \lambda I_{n-m}| = 0 \qquad (4.27)$$

Let

$$A_o = \left[\begin{array}{c|c} A_o(1,1) & A_o(1,2) \\ \hline A_o(2,1) & A_o(2,2) \end{array}\right] \qquad (4.28)$$

then, if there exists a set of parameters that results in $A_o(2,2)$ being stable, then one can set $F = \bar{A}_o(2,2)$, where $\bar{A}_o(2,2)$ is the one particular stable matrix out of the possible variations for $A_o(2,2)$. Then, as was shown before, the transformation matrix Q is given by

$$Q = \left[0 \,|\, I_{n-m}\right] \qquad (4.29)$$

and if

$$F_o = F + T_2 \qquad (4.30)$$

remains stable for all possible T_2 matrices, then the closed-loop system will remain insensitive to parameter variations. The matrix A_1 will be given in this case by

$$A_1 = \left[\begin{array}{c|c} R_1 & 0 \\ \hline T_1 & T_2 \end{array}\right] . \qquad (4.31)$$

The results thus far were developed for continuous time systems; Appendix B presents the extension to discrete-time dynamic systems. An example will illustrate the suggested procedure.

Example: Consider the linear dynamic system described by

$$\dot{x}(t) = A_o x(t) + Bu(t)$$

where

$$A_o = \begin{pmatrix} a & 1 \\ b & -1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

and

$$-3 \leq a \leq 2 \quad , \quad 1 \leq b \leq 4$$

Let

$$A = \begin{pmatrix} -3 & 1 \\ 1 & -1 \end{pmatrix} \quad A_1 = \begin{pmatrix} \Delta a & 0 \\ \Delta b & 0 \end{pmatrix} \qquad \begin{matrix} 0 \leq \Delta a \leq 5 \\ 0 \leq \Delta b \leq 3 \end{matrix}$$

Since $A_4 = -1$, and (A,B) is a controllable pair, we can design a state feedback control law that will accommodate all the variations in the parameters a and b.

Since $\alpha > \Delta a$ is required for the stability of the actual closed-loop system, as established in Eq. (4.), let $\alpha = 6$, then

$$K = -D'(A+\alpha I) = -(1 \quad 0)\begin{pmatrix} 3 & 1 \\ 1 & 5 \end{pmatrix} = (-3 \quad -1) \ .$$

The actual closed-loop matrix is given by

$$\hat{A}_o = A+BK+A_1 = \begin{pmatrix} -6 & 0 \\ 1 & -1 \end{pmatrix} + \begin{pmatrix} \Delta a & 0 \\ \Delta b & 0 \end{pmatrix} = \begin{pmatrix} -6+\Delta a & 0 \\ 1+\Delta b & -1 \end{pmatrix}$$

As seen from the above example, the changes in b do not affect the stability of $\hat{A}_o$, while those of a can be taken care of by a proper choice of the scalar α.

V. CONCLUSIONS

This paper presented the sufficient condition under which the stability properties of the closed-loop system will not be affected by parameter variations in certain submatrices of the system's matrix. Specifically, it was shown that if the variations in the parameter satisfy a certain rule or, if a certain submatrix is stable, a state feedback control law can be found such that the stability properties of the closed-loop system will not be affected by parameter variations.

APPENDIX A

Consider the system described by

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{A.1}$$

and where the nxm matrix B is partioned by

$$B = \begin{bmatrix} B_1 \\ \hline B_2 \end{bmatrix} \begin{matrix} \}m \\ \}n-m \end{matrix} \tag{A.2}$$

Since the matrix B is assumed to be of full rank, one can always renumber the components of the state, if necessary, so that B_1 is an mxm nonsingular matrix.

Let

$$T \triangleq \left[\begin{array}{c|c} B_1^{-1} & 0 \\ \hline -B_2B_1^{-1} & I_{n-m} \end{array}\right] \tag{A.3}$$

if

$$z(t) \triangleq Tx(t) \tag{A.4}$$

then the equivalent system to Eq. (A.1) will be given by

$$\dot{z}(t) = \bar{A}z(t) + \bar{B}u(t) \tag{A.5}$$

where

$$\bar{A} = TAT^{-1} \tag{A.6}$$

$$\bar{B} = TB = \begin{bmatrix} I_m \\ \hline 0 \end{bmatrix} . \tag{A.7}$$

Since Eqs. (A.1 and (A.5) are equivalent systems, one can analyze Eq. (A.5) instead of Eq. (A.1).

The matrix $\bar{A}_{22}$ will be given by

$$\bar{A}_{22} = A_4 - B_2 B_1^{-1} A_2 \ . \tag{A.8}$$

If one solves

$$\bar{Q}\,\bar{A}' - \bar{F}'\,\bar{Q} = \bar{H}\,\bar{B}' \tag{A.9}$$

for $\bar{Q}$, then

$$QA - F'Q = HB' \tag{A.10}$$

is solved too, with

$$\begin{cases} F = \bar{F} & \text{(A.11)} \\ H = \bar{H} & \text{(A.12)} \\ Q = \bar{Q}(T')^{-1} & \text{(A.13)} \end{cases}$$

and

$$\begin{bmatrix} B' \\ \hline Q \end{bmatrix}^{-1} = T' \begin{bmatrix} \bar{B} \\ \hline \bar{Q} \end{bmatrix}^{-1} = T' \left[\begin{array}{c|c} \bar{D} & \bar{E} \end{array} \right] . \tag{A.14}$$

APPENDIX B

The discrete-time dynamic system is described by

$$x(k+1) = A_o x(k) + Bu(k) \tag{B.1}$$

where $x \varepsilon \mathbb{R}^n$, $u \varepsilon \mathbb{R}^m$, and A and B are appropriately dimensioned matrices.

The results reported in section III are applicable to the system given by (B.1). The results of section IV, however, should be slightly modified.

Theorem (4.2) applies to discrete-time dynamic systems except that now (4.21) will require that the scalar α will be chosen such that

$$|\lambda(F_1)| < 1 \tag{B.2}$$

which leads to

$$\lambda(R_1) - 1 < \alpha < \lambda(R_1) + 1. \tag{B.3}$$

The condition expressed in (B.3) may not be satisfied by a single value of α if R_1 has large variations. ■

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APPENDIX D: CONTROL OF FLEXIBLE STRUCTURES

Flexible structures such as an aircraft wing, long beam, or a large space structure (LSS) are representatives from a class of systems with infinite modes. The actual dynamics are governed by partial differential equations for which the lower frequency modes can be adequately (in most cases) modeled by interconnected oscillatory systems. The difficulty arises in that the dimension of this model is extremely large (e.g., thousands of modes), and the higher modal frequencies and modes shapes are poorly known. However, the structure of the model is definitely known--i.e., positive symmetric matrix second order equations. The problem then is to find robust controllers which are based on a lower order model of the flexible structure. Two approaches are presented.

D.1 ROBUST INVERSE OPTIMAL CONTROL FOR FLEXIBLE STRUCTURES

An inverse optimal control problem is formulated to develop robust control laws for purely oscillatory systems. The optimal control solution requires output feedback with specified constraints, leading to robustness with respect to unmodeled modes and a large class of parameter variations. The robustness properties are proved directly from known properties of control laws optimizing quadratic performance indices. These control laws are useful for poorly damped flexible structures.

1. INTRODUCTION

Ever since Kalman (Ref. 1) formulated the inverse optimal control problem in 1964, the subject has been of significant interest in control theory (Refs. 2-5). The inverse optimal control theory attempts to derive a class of performance indices which are optimized by a prespecified control input. In its simplest form, the state and input penalty matrices of a quadratic performance index are determined for linear feedback control of linear dynamic systems (Refs. 1 and 6).

Past research has never exploited the full potential of inverse optimal control theory. Once the control law has been specified, obtaining the optimizing cost functional itself has little direct value. However, when combined with known and previously proven properties of optimal control laws, the derived optimizing function aids the understanding of robustness properties of the feedback control law. An optimizing function is derived for a specific class of systems and is used to study controller properties.

This paper addresses purely oscillatory systems defined by all poles and zeros on the imaginary axis. Flexible systems such as large space structures (LSS) are examples of such systems, because of their typically low damping ratios. Optimizing functions are derived for control laws which increase damping ratios in such systems. The optimizing functions are quadratic. Since the systems are linear, the properties of linear quadratic controllers are used to specify:

(1) robust control structures;

(2) the set of parameter variations for which the controller is robust; and

(3) the class of truncated modes under which the closed loop system is stable.

The inverse optimal control problem considered here simplifies the process of obtaining optimizing functions for a given control law. It is shown, for example, that the procedure for obtaining quadratic cost functions is simplified considerably if the dynamic systems are expressed in modal form. The paper places an emphasis on studying methodologies to utilize the optimizing cost functionals. A system for which these results are directly applicable is that of a large flexible space structure (LSS).

The special forms of dynamic models are given in Section 2. Section 3 presents the problems to be considered. Section 4 solves the inverse optimal control problem and Section 5 presents some further results associated with this solution. Section 6 discusses the important issue of robustness of the closed-loop system.

2. MODEL FORM

The dynamics of continuous flexible structures are described by second order partial differential equations of the wave type. The partial differential equation is not amenable to control design and, therefore, approximate dynamic models are obtained through the application of the finite element method. The dynamic behavior of a flexible system can be closely approximated by the following undamped structural equations

$$M\ddot{\eta} + K\eta = F \tag{1}$$

where η is a vector of generalized displacements at various points on the structure. Equation (1) can be transformed to the following:

$$\ddot{q} + A_o q = B_o u, \quad q \epsilon \mathbf{R}^n, \; u \epsilon \mathbf{R}^m \tag{2}$$

where the $n \times n$ matrix A_o is symmetric, and positive-definite (p.d)

$$A_o = A_o^T > 0 \tag{3}$$

These equations are known as the structural modal equations since the modal structure can be easily derived for the system described in Eq. (2). Defining a state vector

$$x = [q, \dot{q}]^T \tag{4}$$

Eq. (2) can be written in state-space form as

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x \in \mathbb{R}^{2n}, \quad u \in \mathbb{R}^m \tag{5}$$

where

$$A = \left[\begin{array}{c|c} 0 & I_n \\ \hline -A_o & 0 \end{array}\right] \qquad B = \left[\begin{array}{c} 0 \\ \hline B_o \end{array}\right] \tag{6}$$

and B_o is an $n \times m$ matrix of full rank m $(m \leq n)$.

The structural properties of Eqs. (5) and (6) are suitable for the description of the flexible space structure. Therefore, Eqs. (3), through (6) define the modal form to be considered in the inverse control problem.

The purpose of control in flexible systems is to provide damping to the oscillating structure. This is accomplished through energy-dissipating devices. To implement these controls, velocities are measured at different locations on the structure and converted to inputs through actuators (rate feedback). When the sensors and actuators are co-located, the m-dimensional output vector is obtained through

$$y(t) = B^T x(t), \quad y \in R^m \tag{7}$$

where B is the same matrix appearing in Eq. (5).

Let us assume for the moment that the positive-definite (p.d) matrix A_o in Eq. (6) is a diagonal matrix (this can always be done through a simple transformation), i.e.

$$A_o = \mathrm{diag}(a_1, a_2, \ldots, a_n),\ a_i > 0, \forall 1 \le i \le n \tag{8}$$

Then it can be verified (Ref. 7) that the i-jth element of the matrix of right eigenvectors U of the $2n \times 2n$ matrix A in Eq. (6) is given by

$$u_{ij} = \begin{cases} 1 & i = 1,2, \ldots, n \\ & j = 2i-1,\ 2i \\ \lambda_j & i = n+1, \ldots, 2n \\ & j = 2(i-n)-1,\ 2(i-n) \\ 0 & \text{elsewhere} \end{cases} \tag{9}$$

where

$$AU = U\Lambda \tag{10}$$

$$\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_{2n}) \tag{11}$$

$$\lambda_i = \begin{cases} \hat{j}\sqrt{a_{(i+1)/2}} & i = 1,3, \ldots, 2n-1 \\ -\hat{j}\sqrt{a_{i/2}} & i = 2,4, \ldots, 2n \end{cases} \tag{12}$$

and

$$\hat{j} \triangleq \sqrt{-1} \ .$$

Then, using Jacobi's formula for eigenvalues perturbations, it can be shown (Ref. 8) that the use of rate feedback where the sensors and actuators are co-located will result in an eigenvalue shift to the left in the complex plane (increased damping). The control law is an output feedback given by

$$u(t) = Ky(t) = KB^T x(t) \tag{13}$$

This form of direct velocity feedback can be shown to dissipate energy and, therefore, stabilize the system provided that energy in any zero frequency modes is conserved (Refs. 9 and 10) The proof is based on using the energy as a Lyapunov function.

The observation that the closed-loop system, under the conditions described above, is stable motivates the interest in the inverse optimal control problem. The following factors further define closed-loop control requirements.

The dimension of the model described in Eq. (5) depends on the fineness of the grid used in the application of the finite element method. Usually, however, the dimension is quite large and some reduction in order is required to facilitate numerical solutions. This reduction in order will result in modes that are truncated from the final model used for design. The effect of the control derived from the reduced-order model on the truncated modes is of extreme importance (control spillover, Ref.11) The solution of the inverse optimal control problem, to be presented shortly, will be brought to bear on this problem in Section 6.

3. PROBLEM STATEMENT

Consider the linear dynamic system described by

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), & x \varepsilon \mathbf{R}^{2n},\ u \varepsilon \mathbf{R}^m \\ y(t) = B^T x(t), & y \varepsilon \mathbf{R}^m \end{cases} \tag{14}$$

where

$$A = \left[\begin{array}{c|c} 0 & I_n \\ \hline -A_o & 0 \end{array}\right] \qquad B = \left[\begin{array}{c} 0 \\ \hline B_o \end{array}\right] \tag{15}$$

and the $n \times n$ matrix A_o' is symmetric and positive-definite (p.d) i.e.

$$A_o = A_o^T > 0 \tag{16}$$

Next, consider the set of admissible controls, $\mathcal{U}$, defined by

$$\mathcal{U} \overset{\Delta}{=} \{u \varepsilon \mathbf{R}^m : u = Ky, \mathrm{Re}[\lambda(A + BKB^T)] < 0\} \tag{17}$$

where $\lambda(\cdot)$ stands for the eigenvalue of the argument, and $\mathrm{Re}(\cdot)$ is the real part of a complex number.

Let the objective function to be minimized be quadratic in the state and control over an infinite time period, i.e.

$$J(x,u) \overset{\Delta}{=} \int_0^\infty [x^T(t)Qx(t) + u^T(t)Ru(t)]\, dt \tag{18}$$

with

$$Q \geq 0 \;, \quad R > 0 \tag{19}$$

<u>Problem 1:</u> (inverse optimal control problem)

Given an admissible control, u^*, under what conditions on K, Q, and R is this control law optimal?, i.e., when is

$$J(x,u^*) = \min_{u \varepsilon \mathcal{U}} J(x,u) \tag{20}$$

<u>Problem 2:</u>

To what extent, if any, is the solution to the inverse optimal control problem sensitive to truncated modes?

<u>Problem 3:</u>

How are closed-loop stability properties affected by parameter variations (or uncertainty) in system matrices?

4. INVERSE OPTIMAL CONTROL PROBLEM

Part of the problem presented in Section 3 is that of finding for what state and input penalty matrices, Q and R, respectively, a given admissible control also minimizes an LQ problem. If such an optimal control u^* exists, then (Ref. 2)

$$u^* = Ky = -R^{-T}B^{T}Px \tag{21}$$

where P is the $2n \times 2n$ p.d solution of the algebraic Riccati equation (ARE) given by

$$PA + A^{T}P - PBR^{-1}B^{T}P + Q = 0 \tag{22}$$

The inverse optimal control problem is, therefore, the following. Given an admissible rate feedback control law find for what, if any, weight matrices Q and R, the ARE in Eq. (22) is solved.

Let the $2n \times 2n$ symmetric P and Q matrices be partitioned as

$$P = \left[\begin{array}{c|c} P_{11} & P_{12} \\ \hline P_{12}^{T} & P_{22} \end{array}\right] \qquad Q = \left[\begin{array}{c|c} Q_{11} & 0 \\ \hline 0 & Q_{22} \end{array}\right] \tag{23}$$

Then from the ARE in Eq. (22), we find

$$P_{11} - A_{o}P_{22} - P_{12}B_{o}R^{-1}B_{o}^{T}\,P_{22} = 0 \tag{24}$$

$$P_{12}A_{o} + A_{o}P_{12}^{T} + P_{12}B_{o}R^{-1}B_{o}^{T}\,P_{12} = Q_{11} \tag{25}$$

$$P_{22}B_{o}R^{-1}B_{o}^{T}P_{22} - P_{12}P_{12}^{T} = Q_{22} \tag{26}$$

Since a rate feedback control law is used, we find from Eq. (21) that

$$B_o^T P_{12} = 0 \leftrightarrow P_{12} \in \mathcal{N}(B_o^T) \triangleq \{ z \in \mathbf{R}^n : B_o^T z = 0 \} \tag{27}$$

Therefore, P_{12} can be expressed as

$$P_{12} = N_1 \Phi N_1^T \tag{28}$$

where

$$N_1 \in \mathcal{N}(B_o^T) \tag{29}$$

$$\Phi = \Phi^T > 0 \tag{30}$$

The $n \times n$ matrix P_{12} as given in Eq. (28) is symmetric and from Eq. (30), p.s.d. The latter condition is required to satisfy Eq. (25) since $(-A_o)$ is a stable matrix.

Lemma 1:

For a dynamic system under a rate feedback control law, the $n \times n$ matrix P_{12} in Eqs. (24) through (26) has to satisfy

$$P_{12} = 0 \tag{31}$$

Proof:

This is verified by observing Eq. (26) where Q_{22} has to be at least p.s.d. ∎

Since $P_{12} = 0$, the resulting $2n \times 2n$ matrix P that solves the ARE will be a block diagonal matrix. This motivates the following form for the matrix P to be checked as a possible solution of the ARE.

Let

$$P = \alpha B(B^T B)^{-1} B^T + NEN^T \tag{32}$$

where

$$N \epsilon \mathcal{N}(B^T), \quad N \epsilon \mathbf{R}^{2n \times (2n-m)} \tag{33}$$

$$N^T N = I_{2n-m} \tag{34}$$

$$\alpha \epsilon \mathbf{R}^1_+$$

$$E = E^T > 0 \qquad E \epsilon \mathbf{R}^{(2n-m) \times (2n-m)} \tag{35}$$

It is clear that P in Eq. (32) is symmetric. To show that it is also p.d. we note that

$$PB = \alpha B \tag{36}$$

$$PN = NE \tag{37}$$

Therefore, B and N are right eigenvectors of the matrix P corresponding to positive eigenvalues; therefore, P is p.d. Similar results are obtained for left eigenvectors.

Using the form for P as given in Eq. (32) the ARE can be written as

$$PA + A^T P - \alpha^2 BR^{-1} B^T + Q = 0 \tag{38}$$

Since

$$B = \begin{bmatrix} 0 \\ --- \\ B_o \end{bmatrix}$$

we find that to satisfy Eq. (33)

$$N = \left[\begin{array}{c|c} 0 & I_n \\ \hline N_o & 0 \end{array}\right], \quad N_o \epsilon R^{n \times (n-m)} \tag{39}$$

where N_o is chosen such that

$$B_o^T N_o = 0 \leftrightarrow N_o \epsilon \mathcal{N}(B_o^T) \tag{40}$$

$$N_o^T N_o = I_{n-m} \tag{41}$$

Let the symmetric matrix E in Eq. (32) be partitioned as follows

$$E = \begin{array}{c} \left[\begin{array}{c|c} E_1 & E_2 \\ \hline E_2^T & E_3 \end{array}\right] \begin{array}{l} \} \, n-m \\ \} \, n \end{array} \\ \begin{array}{cc} \underbrace{\quad}_{n-m} & \underbrace{\quad}_{n} \end{array} \end{array} \tag{42}$$

<u>Theorem 1:</u>

A necessary and sufficient condition for P in Eq. (32) to solve the ARE in Eq. (22) is that

$$Q = \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & \alpha^2 B_o R^{-1} B_o^T \end{array}\right] \tag{43}$$

$$E = \left[\begin{array}{c|c} \alpha I_{n-m} & 0 \\ \hline 0 & \alpha A_o \end{array}\right] \tag{44}$$

The $2n \times 2n$ p.d. solution matrix P is given by

$$P = \left[\begin{array}{c|c} \alpha A_o & 0 \\ \hline 0 & \alpha I_n \end{array}\right] \tag{45}$$

Proof: (sufficiency)

$$P = \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & \alpha B_o(B_o^T B_o)^{-1} B_o^T \end{array}\right] + \left[\begin{array}{c|c} \alpha A_o & 0 \\ \hline 0 & \alpha N_o N_o^T \end{array}\right]$$

since, from Eqs. (40) and (41)

$$\left[\begin{array}{c} B_o^T \\ \hline N_o^T \end{array}\right]^{-1} = \left[\begin{array}{c|c} B_o(B_o^T B_o)^{-1} & N_o \end{array}\right] \tag{46}$$

we get

$$P = \left[\begin{array}{c|c} \alpha A_o & 0 \\ \hline 0 & \alpha I_n \end{array}\right]$$

$$Q = \alpha^2 BR^{-1}B^T - PA - A^T P = \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & \alpha^2 B_o R^{-1} B_o^T \end{array}\right]$$

(necessity) from Eq. (38)

$$Q = \alpha^2 BR^{-1}B - PA - A P$$

Since

$$P = \left[\begin{array}{c|c} E_3 & E_2^T N_o^T \\ \hline N_o E_2 & \alpha B_o(B_o^T B_o)^{-1} B_o^T + N_o E_1 N_o \end{array}\right]$$

we find

$$
Q = \left[\begin{array}{c|c}
E_2^T N_o A_o + A_o N_o^T E_2 & A_o[\alpha B_o^T (B_o B)^{-1} B_o^T + N_o E_1 N_o^T] - E_3 \\
\hline
[\alpha B_o (B_o^T B_o)^{-1} B_o^T + N_o E_1 N_o^T] A_o - E_3 & \alpha^2 B_o R^{-1} B_o^T - N_o E_2 - E_2 N_o^T
\end{array}\right]
$$

Since Q is block diagonal, then using Eq. (46) and Lemma 1, we find

$$E_1 = \alpha I_{n-m}$$

$$E_2 = 0$$

$$E_3 = \alpha A_o$$

■

Once the solution to the ARE is obtained, one can find the optimal control as well. The case under consideration in this paper is that where the sensors and actuators are co-located. For this system we have the following.

<u>Corollary 1:</u>

When the output feedback control law is chosen as

$$u = Ky \tag{47}$$

where K is a symmetric, $m \times m$ negative-definite (n.d.) matrix and the output is given through

$$y = B^T x \tag{48}$$

then

$$R = -\alpha K^{-1} \tag{49}$$

is the $m \times m$ control penalty matrix of the quadratic objective function

$$J = \int_0^\infty (x^T Q x + u^T R u)\, dt \tag{50}$$

Proof:

The optimal control that minimizes Eq. (50) is given by

$$u^* = -R^{-1} B^T P x$$

using the solution matrix P from Eq. (45) we find

$$u^* = -\alpha R^{-1} B^T x$$

hence

$$K = -\alpha R^{-1}$$

■

5. FURTHER RESULTS

In Section 2, we stated that the closed-loop system with colocated sensors and actuators is stable. The proofs are based on Jacobi's formula for eigenvalue perturbations and energy functions. The solution obtained in the previous section can be used to prove the closed-loop stability properties of the system under the control law derived in Eq. (47).

The stabilization of a linear system by pole-placement via state feedback requires the system given by

$$\dot{x}(t) = Ax(t) + Bu(t)$$

to be fully controllable (Ref. 12) (This requirement can be

relaxed to that of stabilizability (Ref. 13)). This is an important structural property that will be assumed for the system under consideration.

Assumption:

(A,B) is a controllable pair, i.e.

$$\text{Rank}(A,B) = 2n \tag{51}$$

It is well known (Ref. 2) that if $Q = L^T L$, where (A,L) is an observable pair, then the ARE has only one positive-definite solution, P, and the optimal control yields an asymptotically stable closed-loop system. In the case discussed in the previous sections, we have

$$Q = L^T L \tag{52}$$

where the $m \times 2n$ matrix L is given by

$$L = [0,\ \alpha R^{-1/2} B_o^T] \tag{53}$$

We have to show that (A,L) form an observable pair.

Theorem 2:

If (A,B) is a controllable pair, R and A_o are symmetric p.d. matrices and $\alpha \neq 0$ then (A,L) is an observable pair.

Proof:

The controllability matrix is defined by

$$(A,B) \triangleq \left[B \mid AB \mid \cdots \mid A^{2n-1}B\right]$$

and the observability matrix of the pair (A,L) by

$$\mathcal{O}(A,L) \triangleq \left[L^T \,\vdots\, A^T L^T \,\vdots\, \cdots \; (A^T)^{2n-1} L^T \right]$$

then, using the expression for L, we can verify that

$$\mathcal{O}(A,L) = \alpha \left[\begin{array}{c|c} -A_o & 0 \\ \hline 0 & I_n \end{array} \right] \mathcal{C}(A,B) H^{-1/2}$$

where the $2nm \times 2nm$ matrix $H^{-1/2}$ is defined by

$$H^{-1/2} \triangleq \left[\begin{array}{c|c|c|c} R^{-1/2} & 0 & \cdots & 0 \\ \hline 0 & R^{-1/2} & & 0 \\ \hline \vdots & & \ddots & \\ \hline 0 & 0 & & R^{-1/2} \end{array} \right]$$

A simple application of Sylvester's inequality yields the required result. ■

The results obtained thus far can be used to yield a physical interpretation of the solution to the inverse optimal control problem.

Since the state vector, x, is partitioned according to $x = [q,\dot{q}]^T$ the optimal performance is written as

$$J^* = \min_u [J_1 + J_2] \tag{54}$$

where

$$J_1 = \alpha^2 \int_0^\infty q^T B_o R^{-1} B_o^T q \, dt \tag{55}$$

$$J_2 = \int_0^\infty u^T R u \, dt \tag{56}$$

The interpretation suggested above is that the cost functional is the total energy of the system. The term contributed by the state is the kinetic energy where

$$E = \dot{q}^T M \dot{q}$$

and the "mass," M, is given by

$$M = \alpha^{-2} B_o R^{-1} B_o^T \tag{57}$$

Also, the use of velocity feedback where the actuators and sensors are colocated results in an LQ problem where only kinetic energy is weighted.

The results of the inverse optimal control problem, as discussed thus far, are summarized in Table 1.

6. ROBUSTNESS PROPERTIES

As mentioned in Section 2, one of the problems associated with modeling large flexible space structures is that of truncated modes. The matrix describing the dynamics of the flexible system, given by Eq. (6) has eigenvalues distributed on the imaginary axis. The process of mode truncation is carried out by ignoring eigenvalues that are far from the origin on the imaginary axis (high-frequencies). Analytically, the problem of mode truncation is as follows. Let the full order, finite linear system* be described by

$$\dot{z}(t) \triangleq \left[\begin{array}{c} \dot{x}(t) \\ \hline \dot{x}_T(t) \end{array}\right] = \left[\begin{array}{cc|c} \multicolumn{2}{c|}{0} & I \\ \hline -\Lambda_O & 0 & \\ 0 & -\Lambda_T & 0 \end{array}\right] z(t) + \left[\begin{array}{c} 0 \\ \hline B_O \\ \hline B_T \end{array}\right] u(t) \tag{58}$$

* The real flexible system is described through a second order PDE (wave equation). In linear form, the flexible system is of infinite dimension. Equation (58), therefore, considers only a finite number of these modes.

Table 1

Summary

PROBLEM:

$$J^* = \min_{u \in \mathcal{U}} \int_0^\infty (x^T(t)Qx(t) + u^T(t)Ru(t))\, dt$$

s.t.

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x \in \mathbb{R}^{2n}, \quad u \in \mathbb{R}^m$$

$$u(t) = KB^T x(t)$$

$$K = K^T < 0 \qquad \text{(A given gain matrix)}$$

$$A = \left[\begin{array}{c|c} 0 & I_n \\ \hline -A_o & 0 \end{array}\right], \quad B = \left[\begin{array}{c} 0 \\ \hline B_o \end{array}\right], \quad A_o = A_o^T > 0$$

$$x \triangleq [q, \dot{q}]^T, \quad x(0) = x_o$$

INPUT PENALTY MATRIX:

$$R = -\alpha K^{-1}, \quad \alpha > 0$$

STATE PENALTY MATRIX:

$$Q = \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & \alpha^2 B_o R^{-1} B_o^T \end{array}\right]$$

SOLUTION OF THE ALGEBRAIC RICCATI EQUATION (ARE):

$$P = \left[\begin{array}{c|c} \alpha A_o & 0 \\ \hline 0 & \alpha I_n \end{array}\right]$$

OPTIMAL COST:

$$J^* = x_o^T P x_o = \alpha(q_o^T A_o q_o + ||\dot{q}_o||^2)$$

where the subscript T denotes the part of the system to be truncated. The output used for control is given by

$$y(t) = B_o^T x(t) + B_T^T x_T(t), \quad y \in \mathbb{R}^m \tag{59}$$

The matrix A_T contributes to the full order system frequencies higher than those contributed by A_o and, therefore, can be truncated to yield the reduced order model described by

$$\dot{x}(t) = \left[\begin{array}{c|c} 0 & I_n \\ \hline -A_o & 0 \end{array}\right] x(t) + \left[\begin{array}{c} 0 \\ \hline B_o \end{array}\right] u(t) \tag{60}$$

The first robustness result can now be stated.

<u>Theorem 3:</u>

The full order system described in Eq. (58) under the control

$$u(t) = Ky(t), \quad -K = -K^T > 0 \tag{61}$$

and y given by Eq. (59) will remain stable for every A_T provided that (A_o, B_o), (A_T, B_T) are controllable pairs.

<u>Proof:</u>

Using the control as derived from the reduced order system results in the full order system assuming the form given by

$$\dot{z}(t) = \left[\begin{array}{cc|c} \multicolumn{2}{c|}{0} & K \\ \hline -A_o & 0 & \multirow{2}{*}{$\begin{pmatrix} B_o \\ \hline B_T \end{pmatrix} K (B_o : B_T)^T$} \\ \hline 0 & -A_T & \end{array}\right] z(t)$$

Theorems 1 and 2 indicate that this system will be asymptotically stable provided (A_o, B_o) and (A_T, B_T) form controllable pairs. ■

This result shows how the inverse optimal control solution can be put to use to answer the question of control spillover (Ref. 9).

Observing the proof of Theorem 1 we can arrive at the second robustness result. This result shows how stability properties of the closed-loop system are affected by parameter variations in the system matrix A_o.

Corollary 2:

Consider the dynamic system described in Eq. (60) and where the $n \times n$ matrix A_o can have uncertain parameters that belong to a certin set. Applying the output feedback control of Èq. (61) will result in a stable cloosed-loop system for every A_o in the given range of possible parameter variation provided A_o is always symmetric and p.d. over its entire range.

Proof:

From theorem 1.

■

Remark: In the case when A_o is a diagonal $n \times n$ matrix, the above result guarantees stability of the cloosed-loop system provided the diagonal of A_o remains positive over all possible variations.

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D.2 STABILITY OF LQG MODAL CONTROL FOR FLEXIBLE STRUCTURES

The stability of LQG (linear observer based feedback) modal control of large flexible structures is examined. It is shown that the effect of the observer and the spillover (residual modal interaction with primary modes) is equivalent to a finite gain, linear function perturbing an LQ (linear primary modal state feedback) control. Recent results in robustness theory are applied resulting in a frequency domain condition which guarantees stability. The stability condition is applied to the design of a shape/vibration LQG controller using position measurements.

I. INTRODUCTION

Linear-Quadratic-Gaussian (LQG) *control* theory has been proposed (for example, Refs. 1-4) as a methodolgy for designing control systems for large space structurer (LSS). (LQG control is taken here to mean linear observer based feedback). The basic difficulty in controlling an LSS is that the structure is lightly damped (.2 to .5%) and flexible. Thus, the LSS has an infinite number of highly oscillatory modes. An LQG control (or any realistic control) is finite and must be based on a reasonably sized finite representation of the LSS. Consequently, the effect of the unmodeled (residual) modes can cause instability.

This paper examines the stability of LQG control of the LSS. Previous work along these lines (for example, Ref. 5) has resulted in stability bounds, based on Lyapunov Functions, which can be conservative, and which are dependent on a state-space representation of the residual modal system. Unfortunately, knowledge about the residual modes is uncertain and might be more readily available as a transfer function matrix. The main result presented here gives frequency domain conditions, which if satisfied, guaranty asymptotic stability.

It is shown that the effect of the residual modes and the observer are equivalent to placing a finite gain, linear perturbation in the control loop of an LQ (linear quardratic) full state feedback control. Recent results [6-9] in robustness theory for multivariable systems show that there are frequency domain conditions which guaranty stability of control systems in the presence of multiplicative perturbations. The frequency domain conditions can then be applied specifically for LQG modal control of the LSS and can be used for design. The residual system, i.e. the total effect of spillover, is represented as a

transfer function matrix. This form more easily allows for a limited knowledge of the residual modes. For example, knowledge of bounds on the magnitude of the residual system response at different frequencies is sufficient to test for stability.

The paper is organized so that Sections II and III state the basic problem. Section IV outlines the general form of the frequency domain robustness test for the LQS modal control. Section V modifies the test for LQG plus feedthrough and Section VI shows an application, in general form, to a shape/vibration control using position sensors.

II. MODAL REPRESENTATION

Consider a finite element model (FEM) of a lightly damped flexible structure given by,

$$\ddot{\eta} + 2\zeta\Omega\dot{\eta} + \Omega^2\eta = \Phi^T \Gamma u \qquad (2.1)$$

$$q = \Phi\eta$$

where η and q are ℓ-vectors of modal coordinates and generalized displacement coordinates respectively; u is an m-vector of point control forces; Ω is an $\ell \times \ell$ diagnal matrix of modal frequencies $\omega_1 \ldots \omega_\ell$; $\zeta > 0$ is the damping ratio ($\zeta \approx .005$); and Φ is an $n \times \ell$ matrix whose columns are the (approximate) structural mode shapes. Measurements of displacment and velocity are produced by point sensors on the structure. Thus,

$$\begin{aligned} y &= C_D q + C_V \dot{q} \\ &= C_D \Phi\eta + C_V \Phi\dot{\eta} \end{aligned} \qquad (2.2)$$

where y is an m-vector of measured displacements and velocities. In general the dimension of the FEM is very large and equation (2.1) is usually decomposed (by an appropriate ordering of modes) into P-primary modes and R-residual modes, i.e.,

$$\begin{aligned} &\ddot{\eta}_P + 2\zeta_P\Omega_P\dot{\eta}_P + \Omega_P^2\eta_P = \Phi_P^T \Gamma u \\ &\ddot{\eta}_R + 2\zeta_R\Omega_R\dot{\eta}_R + \Omega_R^2\eta_R = \Phi_R^T \Gamma u \\ &y = C_D\Phi_P\eta_P + C_V\Phi_P\dot{\eta}_P + C_D\Phi_R\eta_R + C_V\Phi_R\dot{\eta}_R \end{aligned} \qquad (2.3)$$

where η_P and η_R represent the primary and residual modal coordinate vectors respectively. A more compact representation of (2.3) is:

$$\begin{aligned}
\dot{x}_P &= A_P x_P + B_P u \\
\dot{x}_R &= A_R x_R + B_R u \qquad\qquad (2.4) \\
y &= C_P x_P + C_R x_R
\end{aligned}$$

where $x_P = (\eta_P \ \dot{\eta}_P)^T$, $x_R = (\eta_R \ \dot{\eta}_R)^T$ and

$$A_P = \begin{bmatrix} 0 & I_P \\ -\Omega_P^2 & -2\zeta_P\Omega_P \end{bmatrix}, \quad B_P = \begin{bmatrix} 0 \\ \Phi_P^T B \end{bmatrix}$$

$$A_R = \begin{bmatrix} 0 & I_R \\ -\Omega_R^2 & -2\zeta_R\Omega_R \end{bmatrix}, \quad B_R = \begin{bmatrix} 0 \\ \Phi_R^T B \end{bmatrix} \qquad (2.5)$$

$$C_P = [C_D\Phi_P \quad C_V\Phi_P] , \qquad C_R = [C_D\Phi_R \quad C_V\Phi_R]$$

III. LQG MODAL CONTROL

The LQG modal control [1] is given by,

$$\begin{aligned}
u_{LQG} &= -F\,\hat{x}_P \qquad\qquad (3.1) \\
\dot{\hat{x}}_P &= A_P\hat{x}_P + B_P\,u_{LQG} + K\,(y - C_P\hat{x}_P)
\end{aligned}$$

where (F,K) are selected in accordance with standard LQG Theory [10] based upon (2.4) as if the residual modes had no effect. Of course this omission is only an artifact to select (F,K). The effect of the resdual modes comes through the terms $B_R u$ and $C_R x_R$, designated [1] as control and observation spillover, and may cause degradation of closed-loop performance, if not instability. The problem to be addressed in this paper is to find conditions which guarantee stability of (2.4) using the LQG modal control (3.1).

Define

$$e_p = \bar{x}_p - x_p$$

$$\bar{u}_{LQ} = -F x_p$$

where e_p is the primary mode state error estimate and $\bar{u}_{LQ}$ is the LQ (Linear Quadratic) full state primary modal control which could be used if the primary mode state x_p were available for measurement. The LQG control (3.1) can then be expressed as,

$$u_{LQG} = u_{LQ} - F e_p \qquad (3.3)$$

$$\dot{e}_p = (A_p - KC_p)\, e_p + KC_R\, x_R$$

Direct calculations using Laplace transform as (3.3) and (2.4) yield,

$$u_{LQG}(s) = L(s)\, u_{LQ}(s) \qquad (3.4)$$

where

$$L(s) = (I + E(s))^{-1}$$

$$E(s) = F\Phi_p(s)\, K(I + C_p\Phi_p(s)K)^{-1}\, H_R(s) \qquad (3.5)$$

$$H_R(s) = C_R\Phi_R(s)B_R$$

with

$$\Phi_p(s) = (sI_p - A_p)^{-1}$$

$$\Phi_R(s) = (sI_R - A_R)^{-1} \qquad (3.6)$$

Equation (3.5) can be interpreted to mean that the LQG modal control (3.1) of the modal decomposed system (2.4) is equivalent to a multiplicative perturbation of the idealized LQ modal control. Figure 1 illustrates the block diagram equivalence. Consequently, stability analysis of LQG modal control can be viewed as robustness of LQ modal control to a perturbation represented by the transfer function $L(s)$ in the input channels.

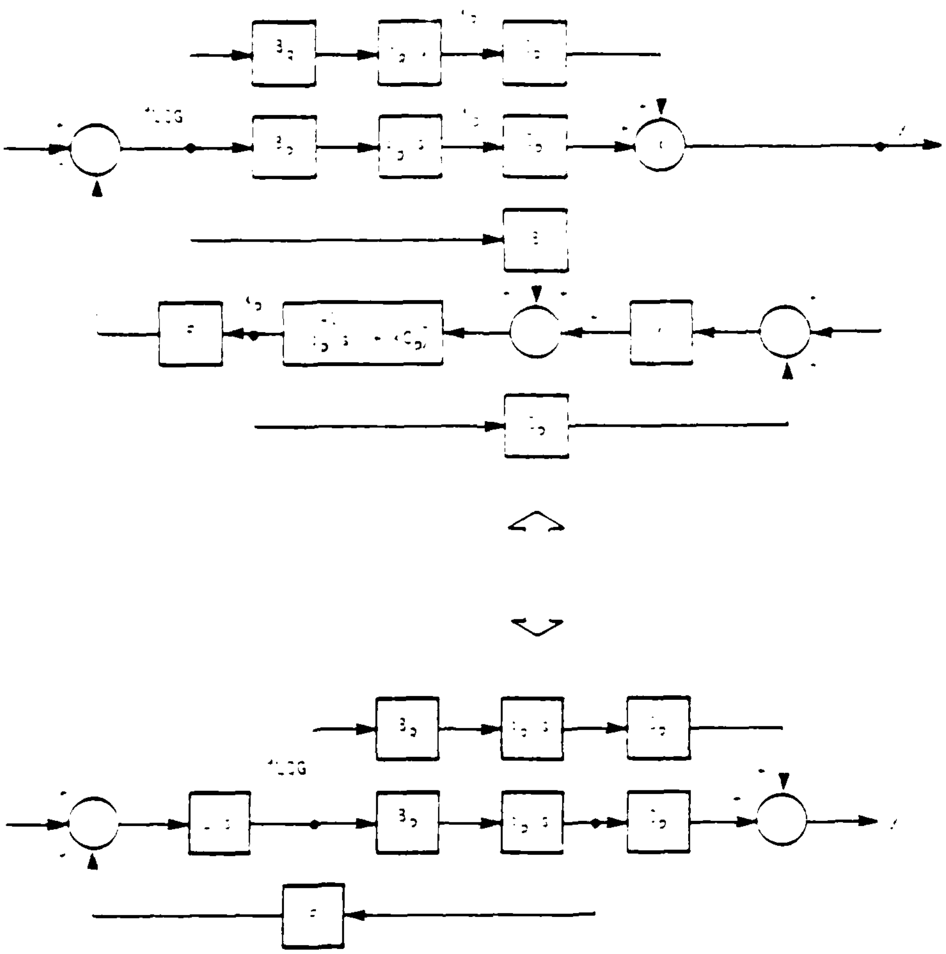

Figure 1 Equivalence of LQG Control and Perturbed LQ Control

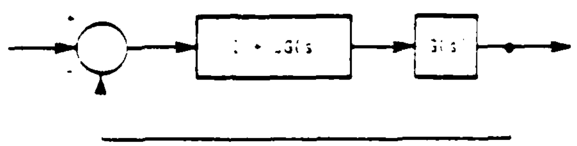

Figure 2 Unity Feedback with Multiplicative Perturbation

IV. ROBUSTNESS TEST FOR STABILITY

Recent studies [6-9] of the stability and robustness of multivariable systems has shown that for the perturbed system of Figure 2, if

$$\underline{\sigma}(I + G^{-1}(j\omega)) > \overline{\sigma}(\Delta G(j\omega)), \quad \omega \geq 0 \qquad (4.1)$$

then the perturbed system is stable. The quantities $\underline{\sigma}(\cdot)$ and $\overline{\sigma}(\cdot)$ are the minimum and maximum singular values respectively, of the matrix argument. Equating the quantities in Figure 2 to Figure 1 gives,

$$G(s) = F\Phi_p(s)B_p \qquad (4.2)$$

$$\Delta G(s) = L(s) - I = -(I + E^{-1}(s))^{-1}$$

since $\overline{\sigma}(M^{-1}) = 1/\underline{\sigma}(M)$ for any matrix M then (4-1) with (4-2) becomes,

$$\underline{\sigma}(I + G^{-1}(j\omega))\ \underline{\sigma}(I + E^{-1}(j\omega)) > 1, \quad \omega \geq 0 \qquad (4.3)$$

Thus, if inequality (4.3) is satisfied then the LQG modal control stabilizes the LSS.

An important aspect of the robustness test (4.3) is that it is a frequency domain test and the spillover effect is contained in $E(s)$ in the term,

$$H_R(s) = C_R\Phi_R(s)B_R \qquad (4.4)$$

This term is the residual input-output transfer function matrix. In a practical sense knowledge of bounds on $H_R(s)$ may be more readily available then the state representation (A_R, B_R, C_R) particularly for the high modal frequencies.

A more conservative test than (4.3) can also be developed. It is shown in [11] that for LQ control,

$$\underline{\sigma}(I + G^{-1}(j\omega)) \geq 1/2 \ , \ \omega \geq 0 \qquad (4.5)$$

consequently (4.3) is surely satisfied if

$$\bar{\sigma}(E(j\omega)) \leq 1, \quad \omega \geq 0 \qquad (4.6)$$

See Appendix A for an alternate derivation based on Ref. 9.

V. LQG WITH DIRECT OUTPUT FEEDBACK

The preceding analysis is also compatible for LQG plus direct output feedback. That is, consider the control,

$$u = - L \hat{x}_p - D y \qquad (5.1)$$

using (2.4), (3.3), and (5.1) results in,

$$\begin{aligned} u &= - L(x_p + e_p) - D(C_p x_p + C_R x_R) \\ &= - (L + DC) x_p - L e_p - D C_R x_R \\ &= u_{LQ} - L e_p - D C_R \ x_R \end{aligned} \qquad (5.2)$$

where

$$u_{LQ} = - F x_p , \ F = L + DC \qquad (5.3)$$

Thus, the residual and error dynamics are,

$$\begin{aligned} \dot{x}_R &= A_{RD} \ x_R - B_R L e_p + B_R u_{LQ} \\ \dot{e}_p &= (A_p - K C_p) e_p + K C_R \ x_R \end{aligned} \qquad (5.4)$$

where

$$A_{RD} = A_R - B_R D C_R \qquad (5.5)$$

According to Canavin [2] the feedthrough gain D is used primarily to influence the residual characteristics A_{RD}.

The perturbation transfer function now becomes (See Appendix B),

$$L_D(s) = (I - D H_{RD}(s)) \ \ (I + Q(s) H_{RD}(s))^{-1} \qquad (5.6)$$

where

$$H_{RD}(s) = C_R \Phi_{RD}(s) B_R$$

$$\Phi_{RD}(s) = (sI_R - A_{RD})^{-1} \tag{5.7}$$

$$Q(s) = L\Phi_p(s)K(I + C_p\Phi_p(s)K)^{-1}$$

If $E_D(s)$ is defined so that,

$$L_D(s) = (I + E_D(s))^{-1} \tag{5.8}$$

then using (25) and (27) results in,

$$E_D(s) = (Q(s) + D)\, G_{RD}(s)(I - DG_{RD}(s))^{-1} \tag{5.9}$$

Following (4.3), if

$$\underline{\sigma}(I + G^{-1}(j\omega))\ \underline{\sigma}(I + E_D^{-1}(j\omega)) > 1,\ \omega \geq 0 \tag{5.10}$$

then LQG plus direct feedthrough stabilizes the system (2.4). Similarly, following (4.6), if

$$\bar{\sigma}(E_D(j\omega)) \leq 1\ ,\ \omega \geq 0 \tag{5.11}$$

the LSS is also stable.

VI. APPLICATION: SHAPE/VIBRATION CONTROL

The robustness test (4.6) will now be applied to an LQG shape and vibration controller. Assume that there are p colocated force actuators and position sensors on the structure. It is desired to control p primary modes using an LQG approach which neglects (for the design) the residual modes. Thus, the primary mode system matrices (A_p, B_p, C_p) are

$$A_p = \begin{bmatrix} 0 & I_p \\ -\Omega_p^2 & -2\zeta_p\Omega_p \end{bmatrix}, \quad C_p = [\Gamma_p^T \; 0] \quad \begin{bmatrix} 0 \\ \Gamma_p \end{bmatrix} \tag{6.1}$$

It is further assumed that B_p has full rank, thus Γ_p^{-1} exists. Partition the controller and observer matrices such that

$$F = [F_1 \; F_2], \quad K = \begin{bmatrix} K_1 \\ K_2 \end{bmatrix} \tag{6.2}$$

Neglecting the residual modes, the characteristic equations for the controller and observer are, respectively

$$\Delta_c(s) = |sI_{2p} - A_p - B_pF| = |s^2 I_p + s(2\zeta_p\Omega_p + \Gamma_pF_2) + \Omega_p^2 + \Gamma_pF_1|$$

$$\Delta_o(s) = |sI_{2p} - A_p + KC_p| = |s^2I_p + s(2\zeta_p\Omega_p + K_1\Gamma^T) + \Omega_p^2 + (K_2 + 2\zeta_p\Omega_p K_1)\Gamma^T| \tag{6.3}$$

Since Γ_p^{-1} exists, it is possible to select (F,K) uniquely so that,

$$\Delta_c(s) = s^2I_p + s2\zeta_c\Omega_c + \Omega_c^2$$

$$\Delta_o(s) = s^2I_p + s2\zeta_o\Omega_o + \Omega_o^2 \tag{6.4}$$

where (ζ_o, ζ_c) are positive scalars and (Ω_c, Ω_o) are diagonal pxp matrices. For the purposes of design, the primary damping ζ_p can be neglected since it is very small. Thus, (6.4) and (6.3) result in,

$$F_1 = \Gamma_p^{-1}(\Omega_c^2 - \Omega_p^2), \qquad F_2 = 2\zeta_c\Gamma_p^{-1}\Omega_c$$

$$K_1 = 2\zeta_o\Omega_o\Gamma_p^{-T}, \qquad K_2 = (\Omega_o^2 - \Omega_p^2)\Gamma_p^{-T} \tag{6.5}$$

After much algebraic manipulation (and using the fact that diagonal matrices commute) the terms in (3.5) become,

$$F\Phi_p(s)K = \Gamma_p^{-1} D^{-1}(s)\, W(s)\Gamma_p^{-T} \tag{6.6}$$

$$I_p + C_p\Phi_p(s)K = \Gamma_p^{T} D^{-1}(s)\, V(s)\Gamma_p^{-T} \tag{6.7}$$

The matrices $D(s)$, $V(s)$, $W(s)$ are all $p \times p$ diagonal and are given by,

$$D(s) = s^2 I_p + \Omega_p^2 \tag{6.8}$$

$$W(s) = 2s(\zeta_o\Omega_o(\Omega_c^2 - \Omega_p^2) + \zeta_c\Omega_c(\Omega_o^2 - \Omega_p^2))$$

$$+ (\Omega_c^2 - \Omega_p^2)(\Omega_o^2 - \Omega_p^2) - 4\zeta_c\zeta_o\Omega_c\Omega_o\Omega_p^2 \tag{6.9}$$

$$V(s) = s^2 I_p + s\, 2\zeta_o\Omega_o + \Omega_o^2 \tag{6.10}$$

Using (6.6) and (6.7) with (3.5) gives

$$E(s) = \Gamma_p^{-1}D^{-1}(s)W(s)\Gamma_p^{-T}[\Gamma_p^{T}D^{-1}(s)V(s)\Gamma_p^{-T}]^{-1}\; H_R(s)$$

$$= \Gamma_p^{-1}D^{-1}(s)W(s)V^{-1}(s)D(s)\Gamma_p^{-T}\; H_R(s)$$

$$= \Gamma_p^{-1}W(s)V^{-1}(s)\Gamma_p^{-T}H_R(s)$$

$$= (\Gamma_p^{T}V(s)W^{-1}(s)\Gamma_p)^{-1}\; H_R(s) \tag{6.11}$$

Consequently, (4.6) is surely satisfied if

$$\bar{\sigma}(H_R(j\omega)) \le \underline{\sigma}^2(\Gamma_p)\underline{\sigma}(V(j\omega)W^{-1}(j\omega)), \qquad \omega \ge 0 \tag{6.12}$$

From (6.8) and (6.9) the diagonal elements of $V(s)$ and $W(s)$ are

$$\left.\begin{aligned} V_i(s) &= s^2 + s\,2\zeta_o\omega_{o_i} + \omega_{o_i}^2 \\ W_i(s) &= a_i s + b_i \end{aligned}\right\} \quad i = 1,2,\ldots,p \tag{6.13}$$

where

$$\begin{aligned} a_i &= 2(\zeta_o\omega_{o_i}(\omega_{c_i}^2 - \omega_{p_i}^2) + \zeta_c\omega_{c_i}(\omega_{o_i}^2 - \omega_{p_i}^2)) \\ b_i &= (\omega_{c_i}^2 - \omega_{p_i}^2)(\omega_{o_i}^2 - \omega_{p_i}^2) - 4\zeta_c\zeta_o\omega_{c_i}\omega_{o_i}\omega_{p_i}^2 \end{aligned} \tag{6.14}$$

Thus,

$$\bar{\sigma}(H_R(j\omega)) \leq \bar{\sigma}^2(\Gamma_p) \min_{i=1\ldots p} \sqrt{\frac{(\omega_{o_i}^2 - \omega^2)^2 + 4\zeta_o^2\omega_{o_i}^2\omega^2}{a_i^2\omega^2 + b_i^2}} \tag{6.15}$$

for all $\omega \geq 0$, guarantees stability.

The left-hand side of (6.15) must be bounded for all frequencies. This requires that the LSS have some natural structural damping. Assuming damping to exist, there is guaranteed stability of the LQG control if the right-hand side is unbounded for all frequencies. This condition would require all a_i and b_i to vanish. One way to achieve this is to eliminate the observer $(K=0)$ and use rate feedback only $(F_1=0,\ F_2=\Gamma^{-1}Q,\ Q=Q^T > 0)$ (see Refs. 12 and 13). This is a limiting case because there is no observer and no shape control, however, (6.15) is validated with respect to a well-known result.

In general, it does appear that (6.15) can be almost always satisfied by some selection of controller/observer gains. The gains are thus constrained and there is a trade-off between the desired behavior of the primary modes and robustness to spillover (residual effects).

Another point worth noting is that the right-hand side of (6.15) is quite accurately known since it is created by design. The left-hand side, which represents the spillover, may not be entirely known for all frequencies. However, knowledge of bounds of the residual transfer function at different frequency bands satisfies the information required in (6.15). For example, given the problem being examined, the residual mode system matrices $(A_R,\ B_R,\ C_R)$ are

$$A_R = \begin{bmatrix} 0 & I_R \\ -\Omega_R^2 & -2\zeta_R\Omega_R \end{bmatrix}, \qquad B_R = C_R^T = \begin{bmatrix} 0 \\ \Gamma_R \end{bmatrix}$$

Consequently,

$$\begin{aligned} H_R(s) &= C_R\Phi_R(s)B_R \\ &= \Gamma_R^T(s^2 I_R + s\ 2\zeta_R\Omega_R + \Omega_R^2)^{-1}\ \Gamma_R \end{aligned}$$

Thus

$$\bar{\sigma}(H_R(j\omega)) \leq \frac{\bar{\sigma}^2(\Gamma_R)}{\displaystyle\min_{i=1\ldots R} \sqrt{(\omega_{R_i}^2 - \omega^2)^2 + 4\zeta_R^2\omega_{R_i}^2\omega^2}} \tag{6.16}$$

Combining (6.16) with (6.15) gives,

$$\begin{aligned} \left(\frac{\bar{\sigma}(\Gamma_R)}{\bar{\sigma}(\Gamma_p)}\right)^4 \leq & \left[\min_{i=1\ldots R}\ (\omega_{R_i}^2 - \omega^2)^2 + 4\zeta_R^2\omega_{R_i}^2\omega^2\right] \\ & \times \left[\min_{i=1\ldots p}\ \frac{(\omega_{o_i}^2 - \omega^2)^2 + 4\zeta_o^2\omega_{o_i}^2\omega^2}{a_i^2\omega^2 + b_i^2}\right] \end{aligned} \tag{6.17}$$

Use of an expression as specific as (6.17) of course requires that (6.16) is accurate, particularly at the high frequencies.

VII. CONCLUSION

The stability of LQG modal control for an LSS has been examined. Results from recent research in robustness theory, gives rise to frequency domain tests for stability. These tests can be used even in the face of uncertainty about the residual modes, particularly the higher order modes.

Future research using the methodology presented here should examine (1) the effect of feedthrough (e.g. colocated active dampers) on the LQG control, (2) the effect of either sensor or controller prefilters, and (3) the effect of sensor and controller dynamics. It also appears that the question of decentralized control of LSS could be examined in this way.

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APPENDIX A
ALTERNATE DERIVATION OF ROBUSTNESS TEST

Following [9] an LQ control will be stability robust for finite-gain, linear input multiplicative perturbations, with transfer function $L(s)$ if

$$L(j\omega) + L^T(-j\omega) \geq I, \quad \omega \geq 0 \tag{A.1}$$

From (3.5)

$$L(s) = (I+E(s))^{-1} \tag{A.2}$$

Using (A.2), (A.1) becomes,

$$(I+E^T(-j\omega))\,[L(j\omega)+L^T(-j\omega)]^T(I+E\,(-j\omega)) \geq (I+E^T(-j\omega))(I+E(j\omega))$$

Thus

$$2I + E^T(-j\omega) = E(j\omega) \geq I+E^T(-j\omega) + E(j\omega) + E^T(-j\omega)\,E(j\omega)$$

and finally

$$E^T(-j\omega)\,E(j\omega) \leq I \tag{A.3}$$

which is implied by

$$\bar{\sigma}(E(j\omega)) \leq 1, \quad \omega \geq 0 \tag{A.4}$$

APPENDIX B
LQG PLUS DIRECT FEEDTHROUGH

To develop (5.6) requires the transfer function between x_R and u_{LQ} and e_p and x_R. Direct calculation from (5.4) gives,

$$x_R = [I + \Phi_{RD}(s)\, B_R Q(s) C_R]^{-1}\, \Phi_{RD}(s) B_R\, u_{LQ} \tag{B.1}$$

$$e_p = (\Phi_p^{-1}(s) + KC_p)^{-1}\, KC_R\, x_R \tag{B.2}$$

where $Q(s)$ and $\Phi_{RD}(s)$ are given by (5.7). From (5.2)

$$\begin{aligned} u_{LQG} &= u_{LQ} - L\, e_p - DC_R\, x_R \\ &= u_{LQ} - (Q(s)+D)C\, x_R \\ &= [I-(Q(s)+D)C_R(I+\Phi_{RD}(s)B_R Q(s)C_R)^{-1}\Phi_{RD}(s)B_R]u_{LQ} \\ &= L_D(s)\, u_{LQ} \end{aligned} \tag{B.3}$$

Using the matrix inversion lemma on $L_D(s)$ gives,

$$\begin{aligned} L_D(s) &= I-(Q+D)C_R[I-\Phi_{RD}B_R(I+Q\, C_R\Phi_{RD}B_R)^{-1}Q\, C_R]\Phi_{RD}B_R \\ &= I-(Q+D)[H_{RD}-H_{RD}(I+Q\, H_{RD})^{-1}Q\, H_{RD}] \\ &= I-(Q+D)H_{RD}(I+Q\, H_{RD})^{-1} \\ &= (I-D\, H_{RD})(I+Q\, H_{RD})^{-1} \end{aligned} \tag{B.4}$$

APPENDIX C
OPEN-LOOP TRANSFER FUNCTION

Given,

$$A = \begin{bmatrix} 0 & I \\ -\Omega^2 & -2\zeta\Omega \end{bmatrix} \tag{C.1}$$

with Ω diagonal. The open-loop transfer function is,

$$\Phi(s) = (sI-A)^{-1} = \begin{bmatrix} (sI+2\zeta\Omega)D^{-1} & D^{-1} \\ -\Omega^2 D^{-1} & s D^{-1} \end{bmatrix} \tag{C.2}$$

where

$$D = s^2 I + s\, 2\zeta\Omega + \Omega^2 \tag{C.3}$$

Consequently,

$$\begin{aligned} F\Phi(s)K = {} & s(F_1 D^{-1} K_1 + F_2 D^{-1} K_2) + F_1 D^{-1} K_2 - F_2 D^{-1} \Omega^2 K_1 \\ & + 2\zeta F_1 D^{-1} K_1 \end{aligned} \tag{C.4}$$

where

$$F = [F_1 \;\; F_2], \quad K = \begin{bmatrix} K_1 \\ K_2 \end{bmatrix} \tag{C.5}$$

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